

TOTALLY NONFREE ACTIONS AND THE INFINITE SYMMETRIC GROUP

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To the memory of my beloved Lyotya

ABSTRACT

We define the notion of a totally nonfree (TNF) action of a group and study its properties. Such an action may be regarded as the adjoint action of the group on the lattice of its subgroups with a special adjoint-invariant measure (TNF measure). The main result is the description of all adjoint-invariant and TNF measures on the lattice of subgroups of the infinite symmetric group $S_{\mathbb{N}}$. The problem is closely related to the theory of characters and factor representations of groups.

1 INTRODUCTION

The main result of this paper is a precise description of all adjoint-invariant ergodic probability Borel measures on the lattice of all subgroups of the infinite symmetric group $S_{\mathbb{N}}$. The reason why problems of this type are of importance can be briefly formulated as follows: the adjoint action of a group on the lattice of subgroups with an adjoint-invariant probability measure produces, in a certain way, a nontrivial character of the group and, consequently, determines a special factor representation of the group.

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In the case of the infinite symmetric group, it turns out that this method yields, up to a small deformation, *all characters of $S_{\mathbb{N}}$* . This phenomenon was first observed in [13], as a particular fact related to a certain model of factor representations of the infinite symmetric group. The list of characters itself was known earlier (Thoma's theorem [8]), but the new proof of this theorem, which used the ergodic approach and approximation suggested in [12], as well as the dynamical approach, made it possible to introduce the so-called groupoid model of factor representations. More precisely, the connection is as follows: the value of an indecomposable character at a given element of the group is equal, up to a certain factor, to the measure of the set of fixed points of this element for some special action. But what actions can appear in this construction? It turns out that these are so-called *totally nonfree* (*TNF*) actions, so it is important to describe such actions for a given group.

In this paper, we go in the opposite direction: first we define and study the class of TNF actions of a group. From the point of view of ergodic theory, this kind of actions is of great interest, and, as far as we know, it has not yet been systematically investigated. Due to the lack of space, we decided to separate the discussion of the link between the questions considered here and the theory of representations, characters, and factors; these topics will be treated in another article.

In the first section, we introduce the main notions and fix definitions related to nonfree and totally nonfree actions. We develop a systematic approach to nonfree actions. Although we are mainly interested in totally nonfree actions, we also consider intermediate cases and the reduction of a general action to a TNF action. The main open problem that arises in this context concerns the *existence of TNF and similar actions for a given group*. We use the language of the *lattice of subgroups and the adjoint action* of the original group on this lattice; as far as we know, the dynamics on this lattice has never been considered from the point of view of ergodic theory. The previous question is equivalent to that of the existence or nonexistence of continuous adjoint-invariant measures. The universal example of a TNF action is the adjoint action of the group on the lattice of its subgroups with a TNF measure, i.e., a measure concentrated on the set of self-normalizing subgroups (= subgroups coinciding with their normalizers). An important result asserts that this measure is a complete metric invariant of the action. In general, there are other adjoint-invariant measures that are not TNF; we describe a procedure (which may be infinite) that produces all TNF measures.

All these facts heavily depend on the properties of the group. It is clear

that for many Lie groups, as well as for groups close to Abelian ones, such a measure does not exist. Of course, for the problem to be interesting, the group should have a continuum of subgroups. For some “large” groups, we have many TNF measures (or many TNF actions), which, fortunately, can be listed up to isomorphism (in contrast to the usual situation in ergodic theory). These measures on the lattice of subgroups can be regarded, in a natural way, as “random subgroups”; for different reasons, the notion of a random subgroup was considered in [6]. The author believes that this question is also of interest within ergodic theory itself.

In the second section, we study the case of the infinite symmetric group. We use a fundamental classical fact about its subgroups; namely, the infinite symmetric group has no primitive subgroups except the alternating subgroup and the whole group itself. This follows from a classical theorem due to C. Jordan (H. Wielandt was perhaps the first to observe this, see [4]). We successively exclude from consideration all other subgroups that cannot lie in the support of an adjoint-invariant measure and reduce the question to a de Finetti-like problem. The description of adjoint-invariant measures on the group $S_{\mathbb{N}}$ relies on the *new notion of a signed Young subgroup*, which is a generalization of the classical notion of a Young subgroup. *It is natural to understand a random signed Young subgroup exactly as a random subgroup whose distribution is an adjoint-invariant measure on $L(S_{\mathbb{N}})$.* The list of parameters α for these measures is exactly the list of Thoma’s parameters. We briefly compare our formula with that of Thoma at the end of the paper; in a sense, our list of adjoint-invariant measures can be regarded as an explanation for the list of characters. We will return to this question elsewhere.

After our short announcement [11] about the concept of a TNF action, it became clear that the papers by R. Grigorchuk and his colleagues [5, 1] contain examples of TNF actions of groups acting on trees. Also, L. Bowen [2] found such examples for the free group. However, an explicit description of the list of all such actions is known only for $S_{\mathbb{N}}$. It turns out that the answer is even closer to the group-theoretic structure; we will apply it to the theory of characters and factor representations in a subsequent paper. Perhaps, our methods can also be applied to other similar groups, such as the group of rational interval exchange, $U(\infty)$ (the infinite unitary group), $GL(F_q, \infty)$, etc.

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2 MAIN DEFINITIONS. TOTALLY NON-FREE ACTIONS

2.1 FIXED POINTS, STABILIZERS, AND SUB- σ -FIELDS

Let (X, \mathfrak{A}, μ) be a Lebesgue space with a probability measure μ defined on a σ -field \mathfrak{A} of classes of mod0 coinciding measurable sets, and let a countable group G act on this space by μ -preserving transformations. We will consider only *effective actions*, which means that only the identity $e \in G$ of the group acts as the identity transformation mod0.

For each element $g \in G$, we define a measurable set X_g called the *set of fixed points* of g :

$$X_g = \{x \in X, \quad gx = x\}.$$

Consider the map

$$\Phi : G \rightarrow \mathfrak{A}; \quad \Phi : g \mapsto X_g.$$

Definition 1. *The fixed point σ -field corresponding to the action of G under consideration is the sub- σ -field \mathfrak{A}_G of the σ -field \mathfrak{A} generated by the family of all sets X_g , $g \in G$.*

The sets X_g are well defined for arbitrary actions of countable groups and, more generally, for *pointwise, or measurable actions* of arbitrary groups.¹ It is worth mentioning that the above definition of the σ -field \mathfrak{A}_G applies to *continuous actions of arbitrary groups*, since the set of fixed points for a given automorphism is well defined with respect to mod0: if $g_1 = g_2 \bmod 0$, then $X_{g_1} = X_{g_2} \bmod 0$.

Remark 1. An action of a group G is called free if $\mu X_g = 0$ for all $g \neq \text{Id}$, $g \in G$, or, in short, if the σ -field \mathfrak{A}_G is trivial (the trivial σ -field will be denoted by \mathfrak{N}).

¹Recall that an action of a group G is called pointwise (or measurable) if there is a measurable set of full measure on which the action of G is defined; an action is called continuous (in Rokhlin's terminology; the other term is mod0-action) if a homomorphism $G \rightarrow \text{Aut}(X, \mu)$ is defined, where $\text{Aut}(X, \mu)$ is the group of all classes of measure-preserving transformations of (X, μ) . For countable groups, as well for locally compact groups, these two notions are equivalent.

For pointwise actions, we can define the notion of the *stationary subgroup*, or the *stabilizer*, of a point $x \in X$:

$$G_x = \{g \in G : gx = x\}.$$

It is clear that if $y = hx$ with $y, x \in X_0$, $h \in G$, then $G_y = h^{-1}G_xh$. In general, this notion is not well defined for uncountable groups; more exactly, it can be defined only if one can introduce the notion of the orbit partition.

Now we are going to define another sub- σ -field of the σ -field \mathfrak{A} in the space X . We start with the following definition.

Definition 2. Consider the partition ξ_G of the space X into the classes of points having the same stabilizer. We call it the *iso-stable partition* of the triple (X, G, μ) .

The iso-stable partition ξ_G is measurable, because it is the limit, over an increasing sequence of finite subsets $K_n \subset G$, $\bigcup_n K_n = G$, of measurable partitions $\xi_G^{K_n}$: $\xi_G = \lim_n \xi_G^{K_n}$, where two points $x, y \in X$ belong to the same block of $\xi_G^{K_n}$ if and only if $K \subset G_x$, $K \subset G_y$. The partition ξ_G is obviously G -invariant, because an element of ξ_G consists of all points that have the same stabilizers.

Definition 3. Let \mathfrak{A}^G be the sub- σ -field of \mathfrak{A} that consists of all sets measurable with respect to the iso-stable partition ξ_G . In the quotient space X/ξ_G , we have a well-defined action of the group G ; it will be called the *reduced action*.

Proposition 1. Assume that there is a pointwise action of a group G on a space (X, μ) with an invariant measure μ . Both sub- σ -fields \mathfrak{A}_G and \mathfrak{A}^G are G -invariant, and the following inclusion holds:

$$\mathfrak{A}_G \subset \mathfrak{A}^G.$$

For a countable group G , both sub- σ -fields coincide:

$$\mathfrak{A}_G = \mathfrak{A}^G \equiv \mathfrak{A}(G).$$

Proof. The first claim is trivial: two points that cannot be separated by their fixed point sets have the same stabilizers. By definition, the σ -field

\mathfrak{A}_G is generated by the family of sets X_g , $g \in G$. But, since the group G is countable, a basis of the σ -field \mathfrak{A}^G consists of the sets

$$Y_K = \bigcap_{g \in K} X_g,$$

where $K \subset \text{Stab}(x) \subset G$ is an arbitrary finite set. Thus the family X_g , $g \in G$, generates both σ -fields in question. \square

For continuous groups, the sub- σ -fields in question do not coincide in general. For instance, considering the action of the orthogonal group $SO(3)$ on the projective space RP_2 , we see that in this case $\mathfrak{A}_G \subsetneq \mathfrak{A}^G$. Indeed, each set of fixed points has zero measure, whence $\mathfrak{A}_G = \mathfrak{N}$ (where \mathfrak{N} is the trivial σ -field), while $\mathfrak{A}^G = \mathfrak{A}$ since the set of all rotations separate the points of P_2R .

2.2 THE LATTICE OF SUBGROUPS AND THE AD-JOINT ACTION

Denote by $L(G)$ the set of all subgroups of a locally compact group G and equip it with the natural weak topology² and the corresponding Borel structure. For a countable group, the space $L(G)$ equipped with this topology is a compact (Cantor) space. The *adjoint action* of the group G on $L(G)$ is defined as follows. Let $g \in G$, $H \in L(G)$; then

$$\text{Ad}(g)H = gHg^{-1}.$$

We will study the dynamical system $(L(G), \text{Ad}(G))$ from the point of view of ergodic theory; namely, we will consider $\text{Ad}(G)$ -invariant Borel measures. The key problem is the existence of continuous (nonatomic) invariant measures.

PROBLEM 1. *For what groups do there exist continuous $\text{Ad}(G)$ -invariant Borel probability measures? Describe all such measures for a given group.*

We will solve this problem for the infinite symmetric group. Of course, the theory we develop here is interesting for countable groups that have uncountably many subgroups.

²A neighborhood of a subgroup in the weak topology is the set of subgroups that have the same intersection with a given compact subset of G . For a discrete group, $L(G)$ is a subspace of the compact space of all subsets of G .

It is known (see [2]) that a non-Abelian free group has a lot of such measures, but one has no general description of these measures. In [5], actions of groups on trees and more general graphs were considered, and it was verified that these actions are TNF.

A natural point of view on $\text{Ad}(G)$ -invariant measures is to regard them as *random subgroups of G* ; more precisely, each $\text{Ad}(G)$ -invariant measure determines a statistics on the set of subgroups, or a random subgroup. The invariance under conjugations is a natural condition for applications. One may refine this condition and consider random subgroups with additional properties (e.g., TNF measures, or $\text{Ad}(G)$ -invariant measures on the set of self-normalizing subgroups, see below). In the recent paper [6], a problem related to random subgroups arises for a different reason.

The lattice structure on the space of subgroups $L(G)$ is a very popular object of algebraic studies (see, e.g., [7]); we will not use it. It is worth mentioning that an important and completely open question concerns the existence of *σ -finite invariant continuous measures on $L(G)$* . As far as we know, ergodic aspects of the natural dynamical system $(L(G), \text{Ad}(G), \nu)$, where ν is an $\text{Ad}(G)$ -invariant measure, has not been seriously studied.

Let us now connect these dynamical systems $(L(G), \text{Ad}(G), \nu)$ with non-free actions of the group G . Namely, we can identify the quotient space with respect to the iso-stable partition ξ_G with $L(G)$.

Definition 4. *Given an action of a group G on a Lebesgue space (X, μ) , consider the map*

$$\Psi : X \rightarrow L(G), \quad \Psi(x) = G_x.$$

It is a measurable homomorphism of the triple (X, G, μ) to the triple $(L(G), \text{Ad}(G), \Psi_\mu)$, where $\Psi_*\mu$ is an $\text{Ad}(G)$ -invariant Borel measure on $L(G)$, the image of the measure μ under Ψ :*

$$\Psi_*(\mu)(B) = \mu\{x : G_x \in B \subset L(G)\}.$$

We will call Ψ_μ the characteristic measure of the action (X, G, μ) .*

Proposition 2. *The characteristic measure $\Psi_*\mu$ is a metric invariant of measure-preserving actions in the following sense: if two measure-preserving actions of a countable group G on spaces (X^i, μ^i) , $i = 1, 2$, are metrically isomorphic, then the corresponding measures $\Psi_*\mu^i$, $i = 1, 2$, on $L(G)$ coincide.*

Proof. It suffices to observe that every isomorphism between two actions of G must send the set of points with a given stabilizer for one action to the same set for the other action. \square

The map Ψ is nothing else than the factorization of the space (X, μ) with respect to the iso-stable partition ξ_G , which identifies the quotient space X/ξ_G with the image $L(G)$. The quotient measure μ_{ξ_G} tautologically coincides with the characteristic measure $\Psi_*\mu$.

For a free action, Ψ is a constant map and the characteristic measure is the δ -measure at the identity subgroup $\{e\} \in L(G)$.

2.3 TOTALLY NONFREE (TNF) ACTIONS

Definition-Theorem 1. *A measure-preserving action of a countable group G on a space (X, μ) is called totally nonfree (TNF) if one of the following equivalent conditions holds:*

1. *The σ -field $\mathfrak{A}_G (= \mathfrak{A}^G = \mathfrak{A}(G))$ generated by all sets of fixed points coincides with the whole σ -field \mathfrak{A} of all measurable subsets of X . Equivalently, the iso-stable partition ξ_G coincides mod 0 with the partition into separate points.*
2. *For almost all (with respect to the measure $\mu \times \mu$) pairs of different points (x, y) , $x \neq y$, their stabilizers do not coincide: $G_x \neq G_y$.*
3. *The map $\Psi : X \rightarrow L(G) = X/\xi_G$ is a mod 0 isomorphism (= separates points mod 0 on (X, μ)). Consequently, the canonical skew product of the action is isomorphic to the action itself.*

If an action is TNF, then we say that its characteristic measure is a TNF measure on $L(G)$.

The equivalence of the above three conditions directly follows from the definitions of the previous section. It is also clear that the definitions are correct with respect to changing the actions on sets of zero measure.

TNF actions are an opposite extreme to free actions.

The metric classification of TNF actions of a countable group G reduces to the calculation of the characteristic measures $\Psi_*\mu$ on the lattice $L(G)$; so the classification problem for TNF actions is, in a sense, smooth (tame), in contrast to the general isomorphism problem in ergodic theory.

In this paper, we will describe the TNF actions for the infinite symmetric group and explain their connection to the representation theory of this group.

Definition 5. The normalizer of a subgroup $\Lambda \subset G$ is the subgroup $N(\Lambda) = \{g \in G : g\Lambda g^{-1} = \Lambda\}$. A subgroup $H \subset G$ for which $N(H) = H$ is called self-normalizing.³ Denote the set of all self-normalizing subgroups of G by $LN(G)$.

The following claim is obvious.

Proposition 3. A transitive action of a group G (the left action of G on a homogeneous space G/H) is totally nonfree if and only if the stabilizer (i.e., H) is a self-normalizing subgroup ($N(H) = H$, or $H \in LN(G)$).

Indeed, any two points belong to the same orbit (for any $x, y \in X$, there exists $h \in G$ such that $y = hx$); if they have the same stabilizer G_x , then $h \in N(G_x)$, where $N(G_x) = \{h \in G : h^{-1}G_x h = G_x\}$ is the normalizer of G_x . Consequently, either $N(G_x) \neq G_x$ and the action is not TNF, or $N(G_x) = G_x$ and the action is TNF.

A similar situation holds for general actions.

Proposition 4. 1. If a measure-preserving action of a group G on a space (X, μ) is TNF, then for almost all $x \in X$ the stabilizers G_x are self-normalizing: $\mu\{x : G_x \in LN(G)\} = 1$.

2. The adjoint action of the group G on the lattice $L(G)$ with an $\text{Ad}(G)$ -invariant TNF measure ν is a TNF action.

3. The adjoint action of the group G on the measure space $(L(G), \nu)$ is TNF if and only if ν -almost all subgroups $H \in L(G)$ are self-normalizing: $\nu(LN(G)) = 1$.

Proof. 1. Assume that the action is TNF, but at the same time there exists a measurable set A of positive μ -measure such that the stabilizer G_x of every point $x \in A$ is not self-normalizing. Then there exists a point $hx \in A$ with $h \in G$, $h \notin G_x$ such that $hx \neq x$ but $hG_x h^{-1} = G_x$; consequently, x and hx have the same stabilizer, which contradicts the TNF property.

2. If ν is a TNF measure on $L(G)$, then, by definition, it is the image $\nu = \Psi_*\mu$, where (X, G, μ) is a space with a TNF action of G . But the action $\text{Ad}(G)$ on $(L(G), \nu)$ is isomorphic to the action of G on the space (X, μ) (see Section 1). At the same time, the TNF property is invariant under isomorphism.

³It is more natural to call such a subgroup *abnormal*, or *anormal*.

3. We have already proved that the condition $\nu(LN(G)) = 1$ is necessary for the action $\text{Ad}(G)$ to be TNF. The converse is obvious since $\text{Stab } H \equiv N(H) = H$. \square

Remark 2. Although, as we have proved, for the action of G on $(L(G), \nu)$, the condition $\nu(LN(G)) = 1$ is equivalent to the TNF property, for a general measure-preserving action of G on a space (X, μ) , the condition $\mu\{x : G_x \in LN(G)\} = 1$ is only necessary but not sufficient for the action to be TNF, because the stabilizers of two points x, y that belong to different orbits can be the same self-normalizing subgroup: $G_x = G_y \in LN(G)$.

Now we can specify Problem 1 formulated above and reduce the description of TNF actions of a group to the following question.

PROBLEM 2. *Given a group G , describe all ergodic continuous (nonatomic) probability TNF measures on the lattice $L(G)$ of its subgroups. Or, equivalently, describe all ergodic continuous (nonatomic) Borel probability $\text{Ad}(G)$ -invariant measures on the subset $LN(G)$ of $L(G)$.*

Perhaps, for some groups (similar to the infinite symmetric group) this problem coincides with Problem 1 concerning the description of all $\text{Ad}(G)$ -invariant measures, but in general these problems are different.

An important problem is to characterize the TNF actions of a given group G from the point of view of ergodic theory and that of representation theory: what ergodic properties can have TNF actions, and what kind of factor representations can arise for TNF actions?

2.4 THE CANONICAL SKEW PRODUCT, AND THE HIERARCHY OF MEASURES ON $L(G)$

It makes sense to consider types of actions intermediate between free and TNF actions, because they also can be studied by the suggested methods. First of all, we consider the factorization with respect to the iso-stable partition ξ_G in order to define the **canonical skew product structure for general actions of groups**.

Consider a measure-preserving action of a countable group G on a Lebesgue space (X, μ) and the G -invariant iso-stable partition ξ_G (see Section 1.1). The reduced action of G on $(X/\xi_G, \mu_{\xi_G})$ (see Definition 3) is isomorphic to the adjoint action of G on the space $L(G)$ equipped with the

characteristic measure $\Psi_*\mu$. We regard the triple $(L(G), \text{Ad}(G), \Psi_*\mu)$ as the base of the canonical skew product structure for the action (X, G, μ) .

Recall that a skew product is defined if we have an action of the group on the base and a 1-cocycle on the base with values in the group of automorphisms of the typical fiber (Y, α) . For a free action of G , this skew product is trivial, the base consists of a single point, and the fiber is $(Y, \alpha) \equiv (X, \mu)$. In the case of a TNF action, Y is a one-point space and the base coincides with the whole space $(X, \mu) \equiv (L(G), \Psi_*\mu)$. In the general case, the action on the base is the adjoint action, and the 1-cocycle c is a measurable function on the product $G \times L(G)$ with values in the group of measure-preserving transformations of the fiber (Y, α) . The general condition on a cocycle c on the space $L(G)$ with the adjoint action of the group G and an arbitrary group of coefficients has the form

$$c(g_1 g_2, H) = c(g_1, H) c(g_2, g_1 H g_1^{-1}), \quad c(e, H) = \text{Id},$$

where $g \in G$, $H \in L(G)$, and $c(\cdot, \cdot)$ is a measurable function on $G \times L(G)$ with values in some group. From this we can conclude that for a fixed H , the restriction of the map $g \mapsto c(g, H)$ to $g \in H$ is a homomorphism of the group H . But the cocycle that defines the canonical skew product has a stronger property.

Proposition 5. *If the action of G on (X, μ) is ergodic, then the above construction defines a decomposition of the space (X, μ) into the direct product $(X \approx L(G) \times Y; \mu_{\xi_G} \times \alpha)$, where (Y, α) is the typical fiber of the skew product; the action of G on the base is the adjoint action with the $\text{Ad}(G)$ -invariant measure $\Psi_*(\mu)$; a 1-cocycle $c(\cdot, \cdot)$ is a function on the space $G \times L(G)$ with values in the group $\text{Aut}(Y, \nu)$ of measure-preserving automorphisms of the space (Y, ν) .*

A necessary and sufficient condition for a cocycle c to define the canonical skew product is as follows.

1. *If ν is a TNF measure, then*

$$c(h, H) = \text{Id}$$

for ν -almost all $H \in L(G)$ and $h \in H$, where Id is the mod 0 identity map on the space Y ; in other words, the homomorphism mentioned above is identical.

2. *If ν is not a TNF measure, then, in addition to the previous condition, the following property holds: for ν -almost all $H \in L(G)$ and $h \in N(H) \setminus H$,*

$$\alpha(\text{Fix}_{c(h, H)}) = 0,$$

where $\text{Fix}_\phi = \{y \in Y : \phi(y) = y\}$ is the set of fixed points of an automorphism ϕ .

The first condition means that for ν -almost all subgroups H , for all $h \in H$, almost all points of Y are fixed points of the automorphism $c(h, H)$; and the second condition means that for ν -almost all subgroups H , for all $h \notin H$, the set of fixed points of $c(h, H)$ is, on the contrary, of zero measure. These conditions on a cocycle follow from that fact that, by definition, the fibers of the partition ξ coincide with the sets of points with a given stabilizer. We will not discuss details and similar questions. Unfortunately, it is difficult to verify whether there exists a cocycle satisfying this condition for a given $\text{Ad}(G)$ -invariant measure ν . At the same time, for the analysis of a given action it is important to use the canonical skew product and to study the corresponding $\text{Ad}(G)$ -invariant measure on $L(G)$ and cocycle.

Now let us consider the base of the canonical skew product.

Definition-Theorem 2. *The action of a group G on a space (X, μ) is called reduced totally nonfree (RTNF) if the reduced action on the base of the canonical skew product (X_{ξ_G}, μ_G) is a TNF action, or, equivalently, $\mu\{x : N(G_x) \in LN(G)\} = 1 \Leftrightarrow \mu\{x : N^2(G_x) = N(G_x)\} = 1$. A RTNF measure on $L(G)$ is, by definition, the characteristic measure ν of a RTNF action: $\nu\{H \in L(G) : N^2(H) = N(H)\} = 1$ (the second normalizer of a subgroup coincides with its first normalizer).*

Recall that the TNF property is equivalent to the condition $\nu\{H \in L(G) : N(H) = H\} = 1$. This follows from the results of the previous section. In general, the quotient of the canonical skew product (i.e., its base) is not a TNF action, because the stabilizer of a point of the quotient space is the normalizer of the stabilizer of the original point, $\text{Stab}_{\Psi(x)} = N(\text{Stab}_x)$, but different stabilizers may have the same normalizers, and, consequently, we can obtain different points with the same stabilizers. Thus we can apply the map Ψ to the base and consider the second canonical skew product of the base, the third one, etc. This gives an hierarchy of Ad -invariant measures on $L(G)$.

The following commutative diagram shows how to continue our classification:

$$\begin{array}{ccc} X & \xrightarrow{\Psi} & L(G) \\ \pi \downarrow & & \downarrow N \\ X/\xi_G & \xrightarrow{\Psi} & L(G). \end{array}$$

Here the map $\Psi : y \mapsto G_y$ associates with a point y its stabilizer, $\pi : X \rightarrow X/\xi_G$ is the canonical projection, and the map $N : H \mapsto N(H)$ associates with a subgroup its normalizer.

Definition 6. *We will denote by $AD(G)$ the space of all $\text{Ad}(G)$ -invariant measures on the lattice $L(G)$ (which will be called in short “AD-measures”), and by CM (“characteristic measures”), the subset of $AD(G)$ consisting of all characteristic measures $\Psi_*\mu = \mu_{\xi_G}$, for an arbitrary triple (X, μ, G) .*

Denote by $\mathcal{N} = N_$ the operation on the set of measures on $L(G)$ corresponding to the normalization of subgroups: $[(\mathcal{N})\nu](F) = \nu(N^{-1}F)$, $F \subset L(G)$. It is clear that \mathcal{N} sends AD-measures to AD-measures.*

It follows from our definitions that if a measure ν on $L(G)$ is RTNF, then the measure $\mathcal{N}(\nu)$ is TNF; in particular, if ν is a TNF measure, then $\mathcal{N}(\nu) = \nu$. Thus the operation $\mathcal{N} : \{\text{RTNF measures}\} \rightarrow \{\text{TNF measures}\}$ is a projection.

We have a hierarchy of AD-measures on the lattice $L(G)$:

$$AD \supset CM \supset \mathcal{N}(CM) \supset \mathcal{N}^2(CM) \supset \cdots \supset RTNF \supset TNF.$$

It is not clear to the author whether the class CM coincides with AD , or, whether an arbitrary AD-measure on $L(G)$ is the characteristic measure for some action; the problem consists in defining an appropriate cocycle for a given AD-measure. I think that for some groups G , the chain of normalizations above can be infinite or even transfinite. But the most interesting classes of measures are the first and the last two ones: TNF, RTNF, CM, and AD.

2.5 REMARKS

1. Fibre bundle over $L(G)$. Each subgroup H is normal subgroup in its normalizer $N(H)$, so we have a fibre bundle over $L(G)$ with a fibre over $H \in L(G)$ — the group $N(H)/H$. This bundle is invariant under the adjoint action of the group. We will use it for the theory of characters of the group.

2. The TNF limit of AD-measures. It is natural to assume that for every measure ν from the class of AD-measures on a given group G , the AD-measure $\cap\nu$ does exist, which is the limit in n of the sequence of successive normalizations: $\nu \mapsto \mathcal{N}_*^n(\nu)$, $n = 1, 2, \dots$. This limit (for some groups, it may be transfinite) must be a TNF measure.

3. TNF actions for continuous groups. As we know, the σ -fields \mathfrak{A}_G and \mathfrak{A}^G can be different. It is natural to define a TNF action of a general group as an action for which the σ -field \mathfrak{A}^G is the complete σ -field, or, for which the stabilizers separate points. In this case, we again have an isomorphism between a TNF action and the adjoint action on the lattice of subgroups.

4. The continuous version of combinatorial multi-transitivity. The continuous counterpart of the notion of transitivity (or topological transitivity) of actions of discrete groups is that of ergodicity. What is the analog of double transitivity? A common explanation is that this is the ergodicity of the action on the Cartesian square. But I believe that this parallel is too weak. The definition of double transitivity in combinatorics can be formulated as the transitivity of the action of the stabilizer of a point x on the space $X \setminus x$. Thus we suggest the notion of multiple transitivity, which is related to our consideration as follows.

Definition 7. *Assume that a countable group G acts on a standard space (X, μ) with a G -invariant continuous measure. We say that the action is double transitive if for μ -almost every point $x \in X$, the action of the stabilizer $G_x \subset G$ on (X, μ) is transitive. We say that the action is k -transitive if for almost every (in the sense of the measure μ^k on X^k) choice of points x_1, x_2, \dots, x_k , the action of the intersection of subgroups $\bigcap_{i=1}^k G_{x_i} L(G)$ on (X, μ) is transitive.*

It is natural to consider this definition only for TNF actions of G . It will be clear that all TNF actions of the infinite symmetric group are k -transitive for an arbitrary positive integer k .

It is of interest to find all countable groups for which TNF k -transitive actions exist for any positive integer k .

This problem is perhaps related to the class of *oligomorphic groups*, which was defined by P. Cameron [3] (a subgroup G of the group $S_{\mathbb{N}}$ of all finite permutations of \mathbb{N} is called oligomorphic if for any positive integer k , the number of orbits of the diagonal action of G in the Cartesian product \mathbb{N}^k is finite).

3 THE LIST OF RANDOM SUBGROUPS OF THE INFINITE SYMMETRIC GROUP

3.1 SIGNED PARTITIONS AND SIGNED YOUNG SUBGROUPS OF SYMMETRIC GROUPS

We consider the countable group $S_{\mathbb{N}}$, the infinite symmetric group of all finite permutations of the set of positive integers \mathbb{N} (or an arbitrary countable set). In this section, we will give the list of all AD-measures on the lattice $L(S_{\mathbb{N}})$ of subgroups of this group and, in particular, the list of TNF measures. We will use some classical facts about permutation groups and the probabilistic approach.

The lattice $L(S_{\mathbb{N}})$ is very large and contains very different types of subgroups. Nevertheless, the support of an AD-measure consists of subgroups of a very special kind: so-called signed Young groups. The topology and the Borel structure on $L(S_{\mathbb{N}})$ are defined as usual; this is a compact (Cantor) space.

Definition 8 (Signed partitions). *A signed partition η of the set \mathbb{N} is a finite or countable partition $\mathbb{N} = \cup_{B \in \mathcal{B}} B$ of \mathbb{N} together with a decomposition $\mathcal{B} = \mathcal{B}^+ \cup \mathcal{B}^- \cup \mathcal{B}^0$ of the set of its blocks, where \mathcal{B}^0 is the set of all single-point blocks; elements of \mathcal{B}^+ are called positive blocks, and elements of \mathcal{B}^- are called negative blocks (thus each positive or negative block contains at least two points), and we denote by B_0 the union of all single-point blocks: $B_0 = \cup_{\{x\} \in \mathcal{B}^0} \{x\}$.*

Denote the set of all signed partitions of \mathbb{N} by $\text{SPart}(\mathbb{N})$.

Recall that in the theory of finite symmetric groups, the Young subgroup Y_{η} corresponding to an ordinary partition $\eta = \{B_1, B_2, \dots, B_k\}$ is $\prod_{i=1}^k S_{B_i}$, where S_B is the symmetric group acting on B . We will define the more general notion of a *signed Young subgroup*, which makes sense both for finite and infinite symmetric groups. We will use the following notation: $S^+(B)$ is the symmetric group of all finite permutations of elements of a set $B \subset \mathbb{N}$, and $S^-(B)$ is the alternating group on B .⁴

⁴Traditionally, the alternating group is denoted by A_n ; V. I. Arnold was very enthusiastic about the idea to denote it by S_n^- in order not to confuse it with the Lie algebra A_n ; I agree with this idea.

Definition 9 (Signed Young subgroups). *The signed Young subgroup Y_η corresponding to a signed partition η of \mathbb{N} is*

$$Y_\eta = \prod_{B \in \mathcal{B}^+} S^+(B) \times \prod_{B \in \mathcal{B}^-} S^-(B).$$

Note that on the set $B_0 \subset \mathbb{N}$, the subgroup Y_η act identically, so that the partition into the orbits of Y_η coincides with η .

It is not difficult to describe the conjugacy class of Young subgroups with respect to the group of inner automorphisms: $Y_\eta \sim Y_{\eta'}$ if and only if η and η' are equivalent up to the action of $S_{\mathbb{N}}$. But it is more important to consider the conjugacy with respect to the group of outer automorphisms. This is the group $S^{\mathbb{N}}$ of all permutations of \mathbb{N} . Denote by r_0^\pm the number of infinite positive (respectively, negative) blocks, and by r_s^\pm the number of finite positive (respectively, negative) blocks of length $s > 1$. Obviously, the list of numbers $\{r_0^\pm, r_1^\pm, \dots\}$ is a complete set of invariants of the group of outer automorphisms.

3.2 STATEMENT OF THE MAIN RESULT

Consider a sequence of positive numbers $\alpha = \{\alpha_i\}_{i \in \mathbb{Z}}$ such that

$$\alpha_i \geq \alpha_{i+1} \geq 0 \text{ for } i > 0; \quad \alpha_{i+1} \geq \alpha_i \geq 0 \text{ for } i < 0; \quad \alpha_0 \geq 0; \quad \sum_{i \in \mathbb{Z}} \alpha_i = 1.$$

Consider a sequence of \mathbb{Z} -valued independent random variables $\xi_n, n \in \mathbb{N}$, with the distribution

$$\text{Prob}\{\xi_n = v\} = \alpha_v \quad \text{for all } n \in \mathbb{N}, v \in \mathbb{Z}.$$

Thus we have defined a Bernoulli measure μ_α on the space of integer sequences

$$\mathbb{Z}^{\mathbb{N}} = \{\xi = \{\xi_n\}_{n \in \mathbb{N}} : \xi_n \in \mathbb{Z}\}.$$

Definition 10 (A random signed Young subgroup and the measures ν_α). *Fix a sequence $\alpha = \{\alpha_i, i \in \mathbb{Z}\}$; for each realization of the random sequence $\{\xi_n\}, n \in \mathbb{N}$, with the distribution μ_α , define a random signed partition $\eta(\xi)$ of \mathbb{N} as follows:*

$$\eta(\xi) = \{B_i \subset \mathbb{N}, i \in \mathbb{Z}\}, \quad B_i := \{n \in \mathbb{N} : \xi_n = i\},$$

where a block is positive (negative) if it has a positive (negative) index and B_0 is understood as the union of one-point blocks. The correspondence $\xi \mapsto \eta(\xi)$ defines a probability measure on the set $\text{SPart}(\mathbb{N})$ of signed partitions, the image of the Bernoulli measure μ_α . Analogously, the correspondence $\xi \mapsto Y_{\eta(\xi)}$ defines a measure, which we denote by ν_α , on the set of signed Young subgroups, i.e., a measure on the lattice $L(S_{\mathbb{N}})$ of subgroups of $S_{\mathbb{N}}$.

It is convenient to denote positive and negative blocks B_i of a signed partition by B_i^+ and B_i^- , respectively. Note that all nonempty blocks of the random signed partition $\eta(\xi)$ that consist of more than one point are infinite with ν_α -probability one.

Theorem 1 (The list of all AD and TNF measures for the group $S_{\mathbb{N}}$). *Every measure ν_α is a Borel ergodic AD-measure on the lattice $L(S_{\mathbb{N}})$; every ergodic probability Borel AD-measure on this lattice coincides with the measure ν_α for some α . The measure ν_α is RTNF, and is TNF if and only if $\alpha_i = 0$ for all $i \leq 0$.*

3.3 PROOFS

Proof. 1. The easy part of the proof is to check that the measures ν_α are indeed ergodic AD-measures on $L(S_{\mathbb{N}})$. The invariance follows from the construction, because μ_α , being a Bernoulli measure, is invariant under all permutations of indices. The symmetric, alternating, and identity subgroups of the symmetric groups $S(B)$ are normal, so they are $\text{Ad}(G)$ -invariant. Consequently, the measure ν_α , being the image of μ_α , is $\text{Ad}(G)$ -invariant. The ergodicity with respect to permutations also follows from the ergodicity of the Bernoulli measure.

2. Now suppose that we have an ergodic AD-measure on $L(S_{\mathbb{N}})$. We will filter out, step by step, classes of subgroups of $S_{\mathbb{N}}$ that cannot support any AD-measure, and will finally obtain the class of signed Young groups as the only possible class. Then we will construct all AD-measures on this class.

a) A classical result of the theory of permutation groups asserts that the group $S_{\mathbb{N}}$ has no primitive subgroups except the whole group $S_{\mathbb{N}}$ and the alternating group.⁵ This is a more or less direct corollary of the fundamental estimates obtained by C. Jordan for finite symmetric groups, which were generalized by H. Wielandt [14] (see, e.g., [4, Chapter 8]). Namely, this is

⁵A primitive subgroup is a subgroup that has no nontrivial invariant partitions.

a corollary of Jordan's theorem asserting that if a primitive subgroup of S_n has an element with support of size k , then $n < \beta(k)$; a sharp bound on $\beta(k)$ is still unknown.

b) Now consider an AD-measure ν on the set of imprimitive but transitive subgroups of $S_{\mathbb{N}}$. Assume that such a subgroup $H \in L(S_{\mathbb{N}})$ has an invariant partition θ . For the action of H to be transitive, all nontrivial minimal blocks of θ must have the same length $l > 1$, which must be finite (because finite permutations cannot move one infinite block to another one).⁶ Denote by $\theta(H)$ the partition of \mathbb{N} into the minimal blocks for H . The map $H \rightarrow \theta(H)$ associates with ν -almost every imprimitive subgroup a partition into blocks of length $l(H)$; because of the $\text{Ad}(G)$ -ergodicity of ν , this length is the same for ν -almost all subgroups H . Thus the map $H \mapsto \theta(H)$ sends ν to a probability measure on the set of partitions of \mathbb{N} with countably many blocks of the same length $l > 1$, and this measure is invariant with respect to the action of $S_{\mathbb{N}}$ on the space of such partitions. Let us show that there are no such finite measures.

Lemma 1. *There are no probability measures on the space $\text{Part}(l)$ of all partitions of \mathbb{N} into (countably many) blocks of length $l > 1$ that are invariant with respect to the group $S_{\mathbb{N}}$.*

Remark. The space $\text{Part}(l)$ equipped with the weak topology is locally compact but not compact; its natural compactification consists of all partitions whose blocks have length at most l .

Proof. Consider the case $l = 2$. Each partition from $\text{Part}(2)$ determines a symmetric matrix $\{a_{i,j}\}$, $a_{i,j} = a_{j,i}$, $a_{i,i} = 0$, $i, j = 1, 2, \dots$, with only one entry in each row and each column equal to 1, all the other entries being equal to 0. But because of the $S_{\mathbb{N}}$ -invariance, the distribution of the entry $a_{i,j}$ does not depend on i, j , $i \neq j$, and must be a probability measure $\{p_0, p_1\}$ on $\{0, 1\}$. By the ergodic theorem,

$$\lim_n \frac{1}{n!} \sum_{g \in S_n} a_{gi,gj} = \bar{a}_{i,j} = 0$$

for all i, j and all matrices $\{a_{i,j}\}$ of this type. Passing to the limit, we obtain $p_0 = p_1 = 0$, which means that a measure on $\text{Part}(2)$ with desired properties

⁶Note that the lengths of all blocks for a given imprimitive group may be either bounded (so-called “almost primitive groups”) or unbounded (“totally imprimitive subgroups”), see [4], but this difference is not important for our purposes.

does not exist.⁷ The only difference in the case $l > 2$ is that we must consider tensors instead of matrices. \square

c) We have proved that an AD-measure on the lattice $L(S_{\mathbb{N}})$ takes the value 0 on the set of all transitive subgroups; so we have reduced the analysis to intransitive subgroups. Fix such a generic intransitive subgroup $H \in L(S_{\mathbb{N}})$ and consider the partition $\eta(H)$ into its transitive components. The action of the group H on each such component must be primitive, because imprimitive cases can be discarded for the same reason as in the previous part of the proof. For the same reason, it is obvious that all components of this partition must be infinite. Consequently, the action of H on each component is either the action of the whole symmetric group, or that of the alternating group (see part a) of the proof), or that of the identity group on the single-point blocks. We denote the blocks by B_i , $i > 0$, when the action of H is the action of the symmetric group of B_i , and by B_i , $i < 0$, when the action of H is the action of the alternating group of B_i . The action of the identity group on all single-point blocks can be regarded as the identity action on the union of such blocks B_0 . Thus we have a *signed partition* $\eta(H)$ such that the action of H on each block B_i , $i > 0$, is the action of $S^+(B_i)$, the action of H on each block B_i , $i < 0$, is the action of $S^-(B_i)$, and the action on B_0 is the identity action. This means that $H \subset Y_\eta = \prod_i S^\pm(B_i)$ and the restriction of the action of H to B_i is the action of $S^\pm(B_i)$. But the measure ν is an AD-measure, so for ν -almost every group H , the orbit of H under conjugation must belong to a set A of full ν -measure. Applying to H the conjugation gHg^{-1} , where $g \in S^\pm(B_{i_0})$, we obtain a subgroup H' which has the same intersection with the product $\prod_{i \neq i_0} S^\pm(B_i)$ as H , whence $H' = S^\pm(B_{i_0}) \prod_{i \neq i_0} S^\pm(B_i)$. Thus if we average the measure δ_H over the orbit of H under conjugation (“ergodic method”), we obtain the same measure ν , and conclude that the set A contains H' . Continuing this process, we see that the set of all ergodic AD-measures on the lattice $L(S_{\mathbb{N}})$ coincides with the set of measures that are the limits of the averages of the δ -measures at signed Young subgroups Y_η .

d) Now we must identify the required measures with the ergodic limits with respect to conjugation of signed Young subgroups. Because of the correspondence between signed Young subgroups and signed partitions, this question is equivalent to the description of $S_{\mathbb{N}}$ -invariant measures on the set of signed partitions. The last question is similar to the classical de Finetti

⁷The limit measure is the δ -measure at the partition of \mathbb{N} into separate points.

problem concerning $S_{\mathbb{N}}$ -invariant measures on the space of all functions on \mathbb{N} (see [9]). The only small difference lies in the fact that, in contrast to the classical situation, here we have three types of blocks of signed partitions instead of one type in the ordinary de Finetti theorem.

Lemma 2 (An analog of de Finetti's theorem). *Consider the space $\text{SPart}(\mathbb{N})$ of signed partitions of \mathbb{N} ; every ergodic $S_{\mathbb{N}}$ -invariant measure on $\text{SPart}(\mathbb{N})$ is determined by a sequence $\alpha = \{\alpha_i\}_{i \in \mathbb{Z}}$ such that $\alpha_i \geq \alpha_{i+1} \geq 0$ for $i > 0$, $\alpha_{i+1} \geq \alpha_i \geq 0$ for $i < 0$, $\alpha_0 \geq 0$, and $\sum_{i \in \mathbb{Z}} \alpha_i = 1$, as described above.*

Proof. The lemma can be proved by any of the methods people use to prove de Finetti's theorem. For completeness, we present a proof, applying our old ergodic method from [9]. In order to find all ergodic measures ν on a compact $S_{\mathbb{N}}$ -space X using the pointwise ergodic theorem for the group $S_{\mathbb{N}}$ (which is in fact a theorem on the convergence of martingales), it suffices to find the weak limits of measures (when they do exist)

$$\lim \frac{1}{n!} \sum_{g \in S_n} \delta_{gx}$$

for all $x \in X$. More exactly, we need to calculate the limits

$$\lim_n \frac{1}{n!} \sum_{g \in S_n} f(gx)$$

for continuous functions $f \in C(X)$. In our case, it suffices to consider cylinder functions on $\text{SPart}(\mathbb{N})$ which depend on finitely many blocks. Fix a signed partition η and label its blocks with integers in an arbitrary way so that positive (negative) integers correspond to positive (negative) blocks and B_0 is the union of one-point blocks. Consider the \mathbb{Z} -valued sequence x_n , $n \in \mathbb{N}$, defined as follows: $x_n = s$ if $n \in B_s$. Now we may say that the signed partition η is the partition corresponding to the sequence $\{x_n\}$, and each such sequence determines a signed partition. The action of $S_{\mathbb{N}}$ on the set of signed partitions and its action by permutations of coordinates of sequences obviously agree, so our problem reduces to the description of $S_{\mathbb{N}}$ -invariant measures on the space of all elements of $\mathbb{Z}^{\mathbb{N}}$. But this is exactly de Finetti's problem. Start with an arbitrary sequence $\{x_n\} \in \mathbb{Z}^{\mathbb{N}}$ and calculate the limit

$$\lim_n \frac{1}{n!} \# \{g \in S_n : x_{gn} = v\} = \alpha_v$$

under the assumption that it does exist (it exists for almost all $x \in X$). Thus we obtain the one-dimensional distribution of the random (with respect to the measure ν) sequence x_n . In order to prove that this measure is a Bernoulli measure on $\mathbb{Z}^{\mathbb{N}}$, we must calculate the joint distribution of several coordinates of x_n . But because of the complete transitivity of the action of S_n , for any choice of v_1, \dots, v_t and for $n \gg t$ we have

$$\frac{\#\{g \in S_n : x_{gi} = v_i, i = 1, 2, \dots, t, n \gg t\}}{n!} \approx \prod_{i=1}^t \alpha_{v_i},$$

which means that the random variables x_n , $n = 1, 2, \dots$, are independent. Thus all AD-measures arise from Bernoulli measures on the space of signed partitions, i.e., $\nu = \nu_\alpha$ for some α . \square

e) Consider the random signed Young subgroup Y_η constructed from a sequence α with $\alpha_i = 0$ for all $i \leq 0$. Then all blocks B_i , $i \leq 0$, are empty with probability one. Then, obviously, the normalizer $N(Y_\eta)$ coincides with Y_η , since each block of η gives rise to the self-normalizing subgroup $S^+(B_i)$. Consequently, the measure ν_α is TNF. If $\alpha_i > 0$ for some $i \leq 0$, then the corresponding block B_i is not empty with probability one, whence $N(S_{B_i}^-) = S_{B_i}^+ \neq S_{B_i}^-$, so that Y_η is not self-normalizing. But

$$N(Y_\eta) = \prod_{i=-\infty}^{+\infty} S_{B_i}^+.$$

Thus $N^2(Y_\eta) = N(Y_\eta)$, so that ν_α is a RTNF measure in the terminology of Section 1. This completes the proof of Theorem 1. \square

Corollary 1. *The action of the group $S_{\mathbb{N}}$ on the measure space $(L(S_{\mathbb{N}}), \nu_\alpha)$ is ergodic.*

Indeed, this is a corollary of the fact that the measure ν_α is the image of the Bernoulli measure μ_α and the correspondence $\mu_\alpha \mapsto \nu_\alpha$ between measures commutes with the action of the group. The corresponding representation of the group $S_{\mathbb{N}}$ in the space $L_{\nu_\alpha}^2(L(S_{\mathbb{N}}))$ will be considered elsewhere.

Corollary 2. *There are three degenerate measures ν_α , in the following cases (in the parentheses we indicate the corresponding character and representation, see below):*

- 1) $\alpha_1 = 1, \alpha_i = 0, i \neq 1$; in this case, $\nu_\alpha = \delta_{S_{\mathbb{N}}} (\chi(g) \equiv 1, \text{ the identity representation})$;
- 2) $\alpha_{-1} = 1, \alpha_i = 0, i \neq -1$; in this case, $\nu_\alpha = \delta_{S_{\mathbb{N}}} (\chi(g) = (-1)^{\text{sgn}(g)}, \text{ the alternating representation})$;
- 3) $\alpha_0 = 1, \alpha_i = 0, i \neq 0$; in this case, $\nu_\alpha = \delta_{\text{Id}_{S_{\mathbb{N}}}} (\chi(g) = \delta_e(g), \text{ the regular representation})$.

An ergodic AD-measure ν_α is atomic only in these three cases (in which it is in fact a δ -measure); in all the other cases, ν_α is a continuous measure.

3.4 REMARKS AND A FORMULA FOR THE MEASURES OF THE SETS OF FIXED POINTS

Make sense to compare the language which we use here (the action on $L(G)$) with that which was used in [13] (the action on the Bernoulli sequences).

More concretely, consider the action of $S_{\mathbb{N}}$ on the space $\mathbb{Z}^{\mathbb{N}}$ (instead of $L(S_{\mathbb{N}})$) and ask for a description of TNF and RTNF measures. The answer is a little bit different than for the action on the space of Young subgroups. Namely, the following result holds.

Proposition 6. *The measure μ_α on the space $X = \mathbb{Z}^{\mathbb{N}}$ with the action of the group $S_{\mathbb{N}}$ is a TNF measure if and only if all $\alpha_i, i \neq 0$, are distinct. If $\alpha_i = \alpha_j$ for some $i \neq j$, then the action of $S_{\mathbb{N}}$ is RTNF but not TNF. The canonical projection $X \rightarrow X/\xi_G$ is the factorization with respect to the following equivalence relation on $X = \mathbb{Z}^{\mathbb{N}}$: two elements $\{x_n\}_{n \in \mathbb{N}}, \{x'_n\}_{n \in \mathbb{N}} \in X$ are equivalent for if for every $v \in \mathbb{Z}$ either $\{n \in \mathbb{N} : x_n = v\} = \{n \in \mathbb{N} : x'_n = v\}$, or there exists $v' \in \mathbb{Z}$ with $\alpha_v = \alpha_{v'}$ such that*

$$\{n \in \mathbb{N} : x_n = v\} = \{n \in \mathbb{N} : x'_n = v'\},$$

and

$$\{n \in \mathbb{N} : x_n = v'\} = \{n \in \mathbb{N} : x'_n = v\}.$$

Thus, in this case the action is RTNF not TNF if we have multiplicity in the values of α : $\alpha_i = \alpha_j, i \neq j$; this is not the case for the action in $L(S_{\mathbb{N}})$.

The supports of the measures ν_α in the topological sense (i.e., the minimal closed subsets of full measure) coincide for all α that have the same number of infinite blocks. The support of ν_α for α having infinitely many infinite blocks coincides with the space of all signed Young subgroups.

In the case of the infinite symmetric group, all AD-measures are invariant under the group $S^{\mathbb{N}}$ of all permutations of \mathbb{N} . The conjugation with respect to this group is an extension of the usual conjugation; but for a generic subgroup H from a set A of full ν_α -measure, its orbit under the action of $S^{\mathbb{N}}$ is much larger than A . In other words, the frequencies α_i are invariant under the usual conjugation, but not under its extension. This fact is related to the so-called Kolmogorov effect (see [10]).

Theorem 1 gives more than just the list of AD-measures on the group $S_{\mathbb{N}}$; it helps to give a new proof of Thoma's formula for indecomposable characters of this group. This will be the subject of our next article, and now we merely carry out the calculations and give a short commentary. Here we present the formula for characters in the "positive" case.⁸

Theorem 2. *For an ergodic AD-measure ν_α ,*

$$\nu_\alpha(F_g) \equiv \nu_\alpha\{H : gHg^{-1} = H\} = \nu_\alpha\{H : g \in N(H)\} = \prod_{n>1} [p_n(\alpha)]^{c_n(g)},$$

where

$$p_n(\alpha) = \sum_{i \neq 0} \alpha_i^n$$

(Newton's power sum) and $c_n(g)$ is the number of cycles of length $n > 1$ of a permutation g .

In the case where $\alpha_i = 0$ for $i < 0$, this formula coincides with Thoma's formula [12] for characters of the infinite symmetric group, because the measure of the set of fixed points is equal to the value of the character:

$$\chi_\alpha(g) = \nu_\alpha(\text{Fix}(g)).$$

In the general case, Thoma's formula involves super-Newton instead of Newton sums:

$$p_n(\alpha) = \sum_{i>0} \alpha_i^n + (-1)^{n-1} \sum_{i<0} \alpha_i^n = \sum_{i \neq 0} (\text{sgn } i)^{n-1} \alpha_i^n.$$

The measure of the set of fixed points does not depend on the types of blocks, but for a general parameter α , the value of the character is not equal just to the measure of this set, the formula involving a certain multiplier (see [13]). We will return to this question and give a model of representations in the next article.

⁸For convenience, we have slightly changed the notation: usually, $\alpha_i \equiv \beta_i$ for $i < 0$, and $\alpha_0 \equiv \gamma$.

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TOTALLY NONFREE ACTIONS AND THE INFINITE SYMMETRIC GROUP

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To the memory of my beloved Lyotya

ABSTRACT

We consider the totally nonfree (TNF) action of a groups and the corresponding adjoint invariant (AD) measures on the lattices of the subgroups of the given group. The main result is the description of all adjoint-invariant and TNF-measures on the lattice of subgroups of the infinite symmetric group $S_{\mathbb{N}}$. The problem is closely related to the theory of characters and factor representations of groups.

1 INTRODUCTION

The main result of this paper is a precise description of all adjoint-invariant ergodic probability Borel measures on the lattice of all subgroups of the infinite symmetric group $S_{\mathbb{N}}$. The reason why problems of this type are of importance can be briefly formulated as follows: the adjoint action of a group on the lattice of subgroups with an adjoint-invariant probability measure produces, in a certain way, a nontrivial character of the group and, consequently, determines a special factor representation of the group.

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In the case of the infinite symmetric group, it turns out that this method yields, *all characters of $S_{\mathbb{N}}$* . This phenomenon was first observed in [16], as a particular fact related to a certain model of factor representations of the infinite symmetric group. The list of characters itself is well-known; E.Thoma posed and gave the first solution of the problem ([9]), the new proof of this theorem, which used the ergodic approach and approximation suggested in [15]. This proof was based on the ideas of the dynamical approach, and approximation of the characters of infinite group with the characters of finite symmetric group. The same dynamical approach, namely, so-called group-poid model, gives the realization of corresponding factor-representations of type II_1 . But in this paper for us more important, that the value of an indecomposable character at a given element of the group is equal, (up to a certain factor), to the measure of the set of fixed points of this element for some special action. The most important thing is that precise link between Thoma's parameters of the characters and parameters of the measure both are the same.

But what kind of the actions can appear in this construction? This is what we define in this paper: it turns out that these are so-called *totally nonfree* (*TNF*) actions, so it is important to describe such actions for a given group.

In this paper, we go in the opposite direction: we start with definition and studying of the class of TNF actions of a group. From the point of view of ergodic theory, this kind of actions is of great interest, and, as far as we know, it has not yet been systematically investigated. Due to the lack of space, we decided to separate the discussion of the link between the questions considered here and the theory of representations, characters, and factors; these topics will be treated in another article.

In the first part of paper (the second section), we introduce the main notions and fix definitions related to nonfree and totally nonfree actions. We develop a systematic approach to nonfree actions. Although we are mainly interested in totally nonfree actions, we also consider intermediate cases and the reduction of a general action to a TNF action. The main open problem that arises in this context concerns the *existence and the list of TNF-actions for a given group*. We use the language of the *lattice of subgroups and the adjoint action* of the original group on this lattice. The previous question is equivalent to that of the existence or nonexistence of continuous adjoint-invariant measures. The universal example of a TNF action is the adjoint action of the group on the lattice of its subgroups with a TNF measure,

i.e., a measure concentrated on the set of self-normalizing subgroups (= subgroups coinciding with their normalizers). An important fact asserts that this measure is a complete metric invariant of the action. In general, there are other adjoint-invariant measures that are not TNF; for example so called RTNF-measures which also produced TNF action. We describe so called canonical skew-product of any action and sequence (which may be infinite or even transfinite) of reduced actions.

All these facts heavily depend on the properties of the group. We consider here only countable groups. Of course, for the problem to be interesting, the group should have a continuum of subgroups. It is clear that for many groups, f.e. for groups close to abelian ones, such a measure does not exist. But for some “large” groups, we have many TNF measures (or many TNF actions), which, fortunately, can be listed up to isomorphism (in contrast to the usual situation in ergodic theory). It is natural to consider adjoint measures on the lattice of subgroups as “random subgroups”; the notion of a random subgroup was considered in [5, 7, 6] but with the different kind of applications. The author believes that this question is also of interest within ergodic theory itself.

In the second part of the paper (the third section), we study the case of the infinite symmetric group. We use a fundamental classical fact about its subgroups; namely, the infinite symmetric group has no primitive subgroups except the alternating subgroup and the whole group itself. This follows from a classical theorem due to C. Jordan (H. Wielandt was perhaps the first to observe this, see [4]). We successively exclude from consideration all other subgroups that cannot lie in the support of an adjoint-invariant measure and reduce the question to a de Finetti-like problem and to Kingman’s theorem about random partitions of the naturals. The description of adjoint-invariant measures on the group $S_{\mathbb{N}}$ relies on the *new important generalization of the classical notion of Young subgroup, - namely, a signed Young subgroup; it is natural to understand a random signed Young subgroup exactly as a random subgroup whose distribution is an adjoint-invariant measure on $L(S_{\mathbb{N}})$* . The list of parameters α for these measures is exactly the list of Thoma’s parameters. We briefly compare our formula with that of Thoma at the end of the paper; in a sense, our list of adjoint-invariant measures can be regarded as an explanation for the list of characters. We will return to this question elsewhere.

The direct proof of the TNF-measures for infinite symmetric group with ergodic method perhaps gives us a new proof of the list of the characters of

this group. The conception presented here partially is based on the paper [16], but the general approach and link to the theory of characters is new one, it was proclaimed firstly in the authors' talk in the Henri Poincare institute [12]. The short announcement of the approach can be found in [14].

Some papers on the different topics are tightly related to our topic: the papers by R. Grigorchuk and his colleagues [1, 2] contain examples of TNF actions of groups acting on trees. Also, papers by L. Bowen [6], found such examples of nontrivial AD-measures for the free group. The papers [5, 7] devoted to IRS = invariant random subgroups or AD-measures on the lattices in our terms but the goals are different.

As it known for author the explicit description of the list of all AD-measures and TNF-actions for the group $S_{\mathbb{N}}$ which we present here, is the first result of his type. Perhaps, this methods can also be applied to other groups similar to $S_{\mathbb{N}}$, such as the group of rational interval exchange, $U(\infty)$ (the infinite unitary group), $GL(F_q, \infty)$, etc. It turns out that our answer is even more tightly connected to group-theoretic structure and to the theory of characters, than it can be assumed before; we will apply it to the theory of characters and factor-representations of $S_{\mathbb{N}}$ and other groups in a subsequent paper.

Professors M.Abort, L.Bowen, Y.Glasner, R.Grigorchuk, Y.Guivarch, N.Gordeev, W.Knapp, T.Nagnibeda-Smirnova, G.Olshansky, M.Zischang gave me the important references on the subject. I am grateful to Natalia Tsilevich for her help in editing of this article.

2 MAIN DEFINITIONS. TOTALLY NON-FREE ACTIONS

2.1 FIXED POINTS, STABILIZERS, AND SUB- σ -FIELDS

Let (X, \mathfrak{A}, μ) be a Lebesgue space with a probability measure μ defined on a σ -field \mathfrak{A} of classes of mod0 coinciding measurable sets, and let a countable group G act on this space by μ -preserving transformations. We will consider only *effective actions*, which means that only the identity $e \in G$ of the group acts as the identity transformation mod0. Because of that we denote by the same letter an element of the group ($g \in G$) and the corresponding

automorphisms ($g : x \mapsto gx$) of the space (X, μ) .

For each element $g \in G$, we define a measurable set X_g called the *set of fixed points* of g :

$$X_g = \{x \in X, \quad gx = x\}.$$

Consider the map

$$\Phi : G \rightarrow \mathfrak{A}; \quad \Phi : g \mapsto X_g.$$

Definition 1. *The fixed point σ -field corresponding to the action of G under consideration is the sub- σ -field \mathfrak{A}_G of the σ -field \mathfrak{A} generated by the family of all sets X_g , $g \in G$.*

The sets X_g are well defined for arbitrary actions of countable groups and, more generally, for *pointwise*, or *measurable actions* of arbitrary groups.¹ It is worth mentioning that the above definition of the σ -field \mathfrak{A}_G applies to *continuous actions of arbitrary groups*, since the set of fixed points for a given automorphism is well defined with respect to mod0: if $g_1 = g_2 \bmod 0$ (as the automorphisms of the space (X, μ)), then $X_{g_1} = X_{g_2} \bmod 0$.

Remark 1. An action of a group G is called free if $\mu X_g = 0$ for all $g \neq \text{Id}$, $g \in G$, or, in short, if the σ -field \mathfrak{A}_G is trivial (the trivial σ -field will be denoted by \mathfrak{N}).

For pointwise actions, we can define the notion of the *stationary subgroup*, or the *stabilizer*, of a point $x \in X$:

$$G_x = \{g \in G : gx = x\}.$$

It is clear that if $y = hx$ with $y, x \in X_0$, $h \in G$, then $G_y = h^{-1}G_xh$. In general, this notion is not well defined for uncountable groups; more exactly, it can be defined only if one can introduce the notion of the orbit partition.

Now we are going to define another sub- σ -field of the σ -field \mathfrak{A} in the space X . We start with the following definition.

¹Recall that an action of a group G is called pointwise (or measurable) if there is a measurable set of full measure on which the action of G is defined; an action is called continuous (in Rokhlin's terminology; the other term is mod0-action) if a homomorphism $G \rightarrow \text{Aut}(X, \mu)$ is defined, where $\text{Aut}(X, \mu)$ is the group of all classes of measure-preserving transformations of (X, μ) . For countable groups, as well for locally compact groups, these two notions are equivalent.

Definition 2. Consider the partition ξ_G of the space X into the classes of points having the same stabilizer. We call it the *iso-stable partition* of the triple (X, G, μ) .

The iso-stable partition ξ_G is measurable, because it is the limit, over an increasing sequence of finite subsets $K_n \subset G$, $\bigcup_n K_n = G$, of measurable partitions $\xi_G^{K_n}$: $\xi_G = \lim_n \xi_G^{K_n}$, where two points $x, y \in X$ belong to the same block of $\xi_G^{K_n}$ if and only if $K \subset G_x$, $K \subset G_y$. The partition ξ_G is obviously G -invariant, because an element of ξ_G consists of all points that have the same stabilizers.

Definition 3. Let \mathfrak{A}^G be the sub- σ -field of \mathfrak{A} that consists of all sets measurable with respect to the iso-stable partition ξ_G . In the quotient space X/ξ_G , we have a well-defined action of the group G with invariant quotient (projection) measure μ_{ξ_G} ; the action of G on $(X/\xi_G, \mu_{\xi_G})$ will be called the *reduced action*.

Proposition 1. Assume that there is a pointwise action of a group G on a space (X, μ) with an invariant measure μ . Both sub- σ -fields \mathfrak{A}_G and \mathfrak{A}^G are G -invariant, and the following inclusion holds:

$$\mathfrak{A}_G \subset \mathfrak{A}^G.$$

For a countable group G , both sub- σ -fields coincide:

$$\mathfrak{A}_G = \mathfrak{A}^G \equiv \mathfrak{A}(G).$$

Proof. The first claim is trivial: two points that cannot be separated by their fixed point sets have the same stabilizers. By definition, the σ -field \mathfrak{A}_G is generated by the family of sets X_g , $g \in G$. But, since the group G is countable, a basis of the σ -field \mathfrak{A}^G consists of the sets

$$Y_K = \bigcap_{g \in K} X_g,$$

where $K \subset \text{Stab}(x) \subset G$ is an arbitrary finite set. Thus the family X_g , $g \in G$, generates both σ -fields in question. \square

For continuous groups, the sub- σ -fields in question do not coincide in general. For instance, considering the action of the orthogonal group $SO(3)$ on the projective space RP_2 , we see that in this case $\mathfrak{A}_G \subsetneq \mathfrak{A}^G$. Indeed, each set of fixed points has zero measure, whence $\mathfrak{A}_G = \mathfrak{N}$ (where \mathfrak{N} is the trivial σ -field), while $\mathfrak{A}^G = \mathfrak{A}$ since the set of all rotations separate the points of P_2R .

2.2 THE LATTICE OF SUBGROUPS AND THE AD-JOINT ACTION

Denote by $L(G)$ the set of all subgroups of a locally compact group G and equip it with the natural weak topology² and the corresponding Borel structure. For a countable group, the space $L(G)$ equipped with this topology is a compact (Cantor) space. The *adjoint action* of the group G on $L(G)$ is defined as follows. Let $g \in G$, $H \in L(G)$; then

$$\text{Ad}(g)H = gHg^{-1}.$$

We will study the dynamical system $(L(G), \text{Ad}(G))$ from the point of view of ergodic theory; namely, we will consider $\text{Ad}(G)$ -invariant Borel measures. The key problem is the existence of continuous (nonatomic) invariant measures.

PROBLEM 1. *For what groups do there exist continuous $\text{Ad}(G)$ -invariant Borel probability measures? Describe all such measures for a given group.*

We will solve this problem for the infinite symmetric group. Of course, the theory we develop here is interesting for countable groups that have uncountably many subgroups.

It is known (see [6]) that a non-Abelian free group has a lot of such measures, but one has no general description of these measures. In [1], actions of groups on trees and more general graphs were considered, and it was verified that these actions are TNF.

A natural point of view on $\text{Ad}(G)$ -invariant measures is to regard them as *random subgroups of G* ; more precisely, each $\text{Ad}(G)$ -invariant measure determines a statistics on the set of subgroups, or a random subgroup. The invariance under conjugations is a natural condition for applications. One may refine this condition and consider random subgroups with additional properties (e.g., TNF measures, or $\text{Ad}(G)$ -invariant measures on the set of self-normalizing subgroups, see below). In the recent paper [7], a problem related to random subgroups arises for a different reason.

The lattice structure on the space of subgroups $L(G)$ is a very popular object of algebraic studies (see, e.g., [8]); we will not use it. It is worth

²A neighborhood of a subgroup in the weak topology is the set of subgroups that have the same intersection with a given compact subset of G . For a discrete group, $L(G)$ is a subspace of the compact space of all subsets of G .

mentioning that an important and completely open question concerns the existence of σ -finite invariant continuous measures on $L(G)$. As far as we know, ergodic aspects of the natural dynamical system $(L(G), \text{Ad}(G), \nu)$, where ν is an $\text{Ad}(G)$ -invariant measure, has not been seriously studied.

Let us now connect these dynamical systems $(L(G), \text{Ad}(G), \nu)$ with non-free actions of the group G . Namely, we can identify the quotient space with respect to the iso-stable partition ξ_G with $L(G)$.

Definition 4. *Given an action of a group G on a Lebesgue space (X, μ) , consider the map*

$$\Psi : X \rightarrow L(G), \quad \Psi(x) = G_x.$$

It is a measurable homomorphism of the triple (X, G, μ) to the triple $(L(G), \text{Ad}(G), \Psi_\mu)$, where $\Psi_*\mu$ is an $\text{Ad}(G)$ -invariant Borel measure on $L(G)$, the image of the measure μ under Ψ :*

$$\Psi_*(\mu)(B) = \mu\{x : G_x \in B \subset L(G)\}.$$

We will call Ψ_μ the characteristic measure of the action (X, G, μ) .*

From definitions it is clear that Ψ is *isomorphism* between the reduced actions of the group G on $(X/\xi_G, \mu_{\xi_G})$ and adjoint action on $(L(G), \Psi_*\mu)$.

Proposition 2. *The characteristic measure $\Psi_*\mu$ is a metric invariant of measure-preserving actions in the following sense: if two measure-preserving actions of a countable group G on spaces (X^i, μ^i) , $i = 1, 2$, are metrically isomorphic, then the corresponding measures $\Psi_*\mu^i$, $i = 1, 2$, on $L(G)$ coincide.*

Proof. It suffices to observe that every isomorphism between two actions of G must send the set of points with a given stabilizer for one action to the same set for the other action. \square

The map Ψ is nothing else than the factorization of the space (X, μ) with respect to the iso-stable partition ξ_G , which identifies the quotient space X/ξ_G with the image $L(G)$. The quotient measure μ_{ξ_G} tautologically coincides with the characteristic measure $\Psi_*\mu$.

For a free action, Ψ is a constant map and the characteristic measure is the δ -measure at the identity subgroup $\{e\} \in L(G)$. If the action of the group is effective then $\bigcap_x G_x = \{e\}$.

2.3 TOTALLY NONFREE (TNF) ACTIONS

Definition-Theorem 1. *A measure-preserving action of a countable group G on a space (X, μ) is called totally nonfree (TNF) if one of the following equivalent conditions holds:*

1. *The σ -field $\mathfrak{A}_G (= \mathfrak{A}^G = \mathfrak{A}(G))$ generated by all sets of fixed points coincides with the whole σ -field \mathfrak{A} of all measurable subsets of X . Equivalently, the iso-stable partition ξ_G coincides mod 0 with the partition into separate points.*
2. *The map $\Psi : X \rightarrow X/\xi_G \simeq L(G)$ is a mod 0 isomorphism mod 0 of the action of G on (X, μ) and adjoint action on $(L(G), \Psi_*\mu)$.*

If an action is TNF, then we say that its characteristic measure is a TNF measure on $L(G)$.

The equivalence of the above two conditions directly follows from the definitions of the previous section. It is also clear that the definitions are correct with respect to changing the actions on sets of zero measure.

TNF actions are an opposite extreme to free actions.

The characteristic measure of the ergodic TNF-action is complete metric invariant therefore the metric classification of TNF actions of a countable group G reduces to the calculation of the characteristic measures $\Psi_*\mu$ on the lattice $L(G)$. Thus the classification problem for TNF actions is, in a sense, smooth (tame), in contrast to the general isomorphism problem in ergodic theory.

Definition 5. *The normalizer of a subgroup $\Lambda \subset G$ is the subgroup $N(\Lambda) = \{g \in G : g\Lambda g^{-1} = \Lambda\}$. A subgroup $H \subset G$ for which $N(H) = H$ is called self-normalizing.³ Denote the set of all self-normalizing subgroups of G by $LN(G)$.*

The following claim is obvious.

Proposition 3. *A transitive action of a group G (the left action of G on a homogeneous space G/H) is totally nonfree if and only if the stabilizer (i.e., H) is a self-normalizing subgroup ($N(H) = H$, or $H \in LN(G)$).*

Indeed, any two points belong to the same orbit (for any $x, y \in X$, there exists $h \in G$ such that $y = hx$); if they have the same stabilizer G_x , then

³It is more natural to call such a subgroup *abnormal*, or *anormal*.

$h \in N(G_x)$, where $N(G_x) = \{h \in G : h^{-1}G_xh = G_x\}$ is the normalizer of G_x . Consequently, either $N(G_x) \neq G_x$ and the action is not TNF, or $N(G_x) = G_x$ and the action is TNF.

A similar situation holds for general actions.

Proposition 4. 1. *If a measure-preserving action of a group G on a space (X, μ) is TNF, then for almost all $x \in X$ the stabilizers G_x are self-normalizing: $N(G_x) = G_x$; or $\mu\{x : G_x \in LN(G)\} = 1$. In other words, the characteristic measure of the TNF-action is concentrated on the set $LN(G)$ of self-normalizing subgroups.*

2. *The adjoint action of the group G on the measure space $(L(G), \nu)$ is TNF if and only if ν -almost all subgroups $H \in L(G)$ have self-normalizing normalizer: $N(N(H)) \equiv N^2(H) = N(H)$. In particular, the adjoint action of the group on the lattice $(L(G), \nu)$ with an $\text{Ad}(G)$ -invariant TNF-measure ν is a TNF action.*

Proof. 1. Assume that the action is TNF, but at the same time there exists a measurable set A of positive μ -measure such that the stabilizer G_x of every point $x \in A$ is not self-normalizing. Then there exists a point $hx \in A$ with $h \in G$, $h \notin G_x$ such that $hx \neq x$ but $hG_xh^{-1} = G_x$; consequently, x and hx have the same stabilizer, which contradicts the TNF property.

2. For adjoint action of the group G on $L(G)$ the stabilizer $G_H = N(H)$, so condition $N(G_H) = G_H$ is equivalent to the condition $N(H) = N^2(H)$ for ν -almost all H ; by the item 1 we have TNF-action. \square

Remark 2. 1. *As we saw the adjoint action on $L(G), \nu$ could be TNF-action not only for TNF-measures but for ν with property $\nu\{H : N^2(H) = H\} = 1$. We will call this measures RTNF-measures. In other words - TNF adjoint action takes place for measures ν which are concentrated on $LN^2(G) : \nu(LN^2(G)) = 1$. We will see that for infinite symmetric group we have the examples of those measures.*

2. *The condition $\mu\{x : G_x \in LN(G)\} = 1$ is only necessary but not sufficient for the action to be TNF, because the stabilizers of two points x, y that belong to different orbits can be the same self-normalizing subgroup: $G_x = G_y \in LN(G)$.*

Now we can specify Problem 1 formulated above and reduce the description of TNF actions of a group to the following question.

PROBLEM 2. *Given a group G , describe all ergodic continuous (nonatomic) probability TNF (correspondingly RTNF) measures on the lattice $L(G)$ of its subgroups. Or, equivalently, describe all ergodic continuous (nonatomic) Borel probability $\text{Ad}(G)$ -invariant measures on the subset $LN(G)$ (correspondingly on $LN^2(G)$) of whole space $L(G)$.*

We will see the different answer on the Problem 1 and Problem 2 for infinite symmetric group.

Remark that for ergodic TNF-measure ν the zero-one law with respect to $LN(G)$ takes place: either $\nu(LN(G)) = 0$ or $\nu(LN(G)) = 1$. It is interesting to characterize the TNF actions of a given group G from the point of view of ergodic theory and that of representation theory: what kind of ergodic properties can have TNF actions, and what kind of factor representations can arise for TNF actions? etc.

It is interesting also to describe other classes of $\text{Ad}(G)$ -invariant measures depending on the property of subgroup of full measure (or to describe random subgroup of the various algebraic types).

2.4 THE CANONICAL SKEW PRODUCT, AND REDUCED (RTNF) ACTION.

Now we consider the general actions and describe the canonical reduction which leads to a TNF actions. First of all, we consider the factorization with respect to the iso-stable partition ξ_G in order to define the **the first canonical skew product structure for general actions of groups**.

Consider a measure-preserving action of a countable group G on a Lebesgue space (X, μ) and the G -invariant iso-stable partition ξ_G (see Section 1.1). The reduced action (quotient action) of G on the space $(X/\xi_G, \mu_{\xi_G})$ (see Definition 3) is isomorphic to the adjoint action of G on the space $L(G)$ equipped with the characteristic measure $\Psi_*\mu$. We regard the triple $(L(G), \text{Ad}(G), \Psi_*\mu)$ as the base of the canonical skew product structure for the action (X, G, μ) .

Recall that a skew product is defined if we have an action of the group on the base and a 1-cocycle on the base with values in the group of automorphisms of the typical fiber (Y, α) . For a free action of G , this skew product is trivial, the base consists of a single point, and the fiber is $(Y, \alpha) \equiv (X, \mu)$. In the case of a TNF action, Y is a one-point space and the base coincides with the whole space $(X, \mu) \equiv (L(G), \Psi_*\mu)$.

Definition 6. *The action of the group G on the base $(X/\xi_G, \mu_{\xi_G}) \simeq (L(G), \Psi_*\mu, \text{Ad}(G),)$ we call canonical reduced action.*

The action on the space (X, μ, G) becomes a skew-product with fiber (Y, α) , and we have the 1-cocycle c which is a measurable function on the product of the group and base with values in the group of measure-preserving transformations of the fiber (Y, α) : $c : G \times L(G) \rightarrow \text{Aut}Y$. Recall that the general condition on a cocycle c on the space $L(G)$ with the adjoint action of the group G and an arbitrary group of coefficients has the form

$$c(g_1g_2, H) = c(g_1, H)c(g_2, g_1Hg_1^{-1}), \quad c(e, H) = \text{Id},$$

where $g \in G$, $H \in L(G)$, and $c(\cdot, \cdot)$ is a measurable function on $G \times L(G)$ with values in some group. From this we can conclude that for a fixed H , the restriction of the map $g \mapsto c(g, H)$ to $g \in H$ is a homomorphism of the group H . But our cocycle that defines the canonical skew product has a stronger property.

Proposition 5. *If the action of G on (X, μ) is ergodic, then the above construction defines a decomposition of the space (X, μ) into the direct product $(X \approx L(G) \times Y; \mu_{\xi_G} \times \alpha)$, where (Y, α) is the typical fiber of the skew product; the action of G on the base is the adjoint action with the $\text{Ad}(G)$ -invariant measure $\Psi_*(\mu)$; a 1-cocycle $c(\cdot, \cdot)$ is a function on the space $G \times L(G)$ with values in the group $\text{Aut}(Y, \nu)$ of measure-preserving automorphisms of the space (Y, ν) .*

A necessary and sufficient condition for a cocycle c to define the canonical skew product is as follows.

1. *If ν is a TNF measure, which means that action on the base is TNF-action, then*

$$c(h, H) = \text{Id}$$

for ν -almost all $H \in L(G)$ and $h \in H$, where Id is the mod 0 identity map on the space Y ; in other words, the homomorphism mentioned above is identical.

2. *If ν is not a TNF measure, then, in addition to the previous condition, the following property holds: for ν -almost all $H \in L(G)$ and $h \in N(H) \setminus H$,*

$$\alpha(\text{Fix}_{c(h, H)}) = 0,$$

where $\text{Fix}_\phi = \{y \in Y : \phi(y) = y\}$ is the set of fixed points of an automorphism ϕ .

The first condition means that for ν -almost all subgroups H , for all $h \in H$, almost all points of Y are fixed points of the automorphism $c(h, H)$; and the second condition means that for ν -almost all subgroups H , for all $h \notin H$, the set of fixed points of $c(h, H)$ is, on the contrary, of zero measure. These conditions on a cocycle follow from that fact that, by definition, the fibers of the partition ξ coincide with the sets of points with a given stabilizer. We will not discuss details and similar questions. Unfortunately, it is difficult to verify whether there exists a cocycle satisfying this condition for a given $\text{Ad}(G)$ -invariant measure ν . At the same time, for the analysis of a given action it is important to use the canonical skew product and to study the corresponding $\text{Ad}(G)$ -invariant measure on $L(G)$ and cocycle.

Now let us consider the action of the group G on the base of the canonical skew-product. We assume that this action is effective, in opposite case we must apply all arguments to the quotient group of group G over kernel.

Definition-Theorem 2. *Let $\pi : X \rightarrow (X/\xi_G)$ -a canonical projection over iso-stable partition; the stabilizer of the projection of the point with respect to reduced action is normalizer of the stabilizer:*

$$\text{Stab}_{\pi(x)} = N(\text{Stab}_x).$$

The action of a group G on a space (X, μ) is called reduced totally nonfree (RTNF) if the reduced action is a TNF action, or, equivalently, (see section above) $\mu\{x : N(G_x) \in \text{LN}(G)\} = 1 \Leftrightarrow \mu\{x : N^2(G_x) = N(G_x)\} = 1$.

The adjoint action of the element of $g \in \text{Stab}_{\pi(x)}$ must preserve the stabilizer of Stab_x by construction, this means that $g \in N(\text{Stab}_x)$ and and and vice versa. A RTNF-measure on $L(G)$ is, by definition, the characteristic measure ν of a RTNF action and has property: $\nu\{H \in L(G) : N^2(H) = N(H)\} = 1$ (the second normalizer of a subgroup coincides with its first normalizer).

The following commutative diagram shows the first step of our classification:

$$\begin{array}{ccc} X & \xrightarrow{\Psi} & L(G) \\ \pi \downarrow & & \downarrow N \\ X/\xi_G & \xrightarrow{\Psi} & L(G). \end{array}$$

Here the map $\Psi : y \mapsto G_y$ associates with a point y its stabilizer, $\pi : X \rightarrow X/\xi_G$ is the canonical projection, and the map $N : H \mapsto N(H)$ associates with a subgroup its normalizer.

In general, the quotient of the canonical skew product is not a TNF action, because the stabilizer of a point of the quotient space is the normalizer of the stabilizer of the original point, $\text{Stab}_{\Psi(x)} = N(\text{Stab}_x)$, but different stabilizers may have the same normalizers, and, consequently, we can obtain different points with the same stabilizers. Thus we can apply again the map Ψ to the base $(X/\xi_G, \nu_\xi)$ and consider the second canonical skew product of the base, the third one, etc. This gives an hierarchy of Ad-invariant measures on $L(G)$.

Definition 7. *We will denote by $AD(G)$ the space of all $\text{Ad}(G)$ -invariant measures on the lattice $L(G)$ (which will be called in short “AD-measures”). Denote by $\mathcal{N} = N_*$ the operation on the set of measures on $L(G)$ corresponding to the normalization of subgroups: $[(\mathcal{N})\nu](F) = \nu(N^{-1}F)$, $F \subset L(G)$. It is clear that \mathcal{N} sends AD-measures to AD-measures.*

It follows from our definitions that if a measure ν on $L(G)$ is RTNF, then the measure $\mathcal{N}(\nu)$ is TNF; in particular, if ν is a TNF measure, then $\mathcal{N}(\nu) = \nu$. Thus the operation $\mathcal{N} : \{\text{RTNF measures}\} \rightarrow \{\text{TNF measures}\}$ is a projection.

We have a hierarchy of AD-measures on the lattice $L(G)$:

$$AD \supset \mathcal{N}(AD) \supset \mathcal{N}^2(AD) \supset \dots \supset RTNF \supset TNF.$$

It is natural to assume that for some groups G , the chain of these normalizations as well as the chain of the steps of reductions above can be infinite or even transfinite. The most interesting classes of AD-measures TNF, RTNF, and AD itself.

Remark that for RTNF measure ν the measure $\mathcal{N}(\nu)$ is TNF measure, and, although ν is not TNF-measure, the adjoint action of the group G on the $(L(G), \nu)$ is TNF-action. Indeed, by definition of RTNF for ν -almost all subgroup H , $N^2(H) = N(H)$, but $N(H)$ is the stabilizer of H , so ν -almost all stabilizers a self-normal. Moreover, adjoint action of G on the space $(L(G), \nu)$ for RTNF-measure ν is metrically isomorphic the adjoint action of G on the space $(L(G), \mathcal{N}(\nu))$ and normalization $N : L(G) \rightarrow L(G)$ is that isomorphism of the spaces and actions.

2.5 REMARKS

1. AD-measures The natural question -is it true that each ergodic AD-measures is characteristic measure for some ergodic action of the group G .

We formulate the necessary and sufficient condition on *AD* measure to be characteristic.

Proposition 6. *Suppose ν is ergodic AD-measure on $L(G)$; ν is characteristic measure for an ergodic action of G on a space (X, μ) iff there exist probability AD-measure $\bar{\nu}$ with properties:*

1) *adjoint action of G on $(L(G), \bar{\nu})$ is ergodic;*

2) $\mathcal{N}(\bar{\nu}) = \nu$

In this case we can define $X = L(G), \mu = \bar{\nu}$.

It is not clear if such a measure $\bar{\nu}$ exists for all *AD*-measures ν .

2. Fibre bundle over $L(G)$. Each subgroup H is normal subgroup in its normalizer $N(H)$, so we have a fibre bundle over $L(G)$ with a fibre over $H \in L(G)$ — the group $N(H)/H$. This bundle is invariant under the adjoint action of the group. We will use it for the theory of characters of the group.

3. The TNF limit of the normalizations of AD-measures. It is natural to assume that for every measure ν from the class of AD-measures on a given group G , the AD-measure $\cap \nu$ does exist, which is the limit in n of the sequence of successive normalizations: $\nu \mapsto \mathcal{N}_*^n(\nu)$, $n = 1, 2, \dots$. This limit (for some groups, it may be transfinite) must be a TNF measure.

4. TNF actions for continuous groups. As we know, the σ -fields \mathfrak{A}_G and \mathfrak{A}^G can be different. It is natural to define a TNF action of a general group as an action for which the σ -field \mathfrak{A}^G is the complete σ -field, or, for which the stabilizers separate points. In this case, we again have an isomorphism between a TNF action and the adjoint action on the lattice of subgroups.

5. The continuous version of combinatorial multi-transitivity. The continuous counterpart of the notion of transitivity (or topological transitivity) of actions of discrete groups is that of ergodicity. What is the analog of double transitivity? A common explanation is that this is the ergodicity of the action on the Cartesian square. But I believe that this parallel is too weak. The definition of double transitivity in combinatorics can be formulated as the transitivity of the action of the stabilizer of a point x on the space $X \setminus x$. Thus we suggest the notion of multiple transitivity, which is related to our consideration as follows.

Definition 8. *Assume that a countable group G acts on a standard space (X, μ) with a G -invariant continuous measure. We say that the action is*

metrically double transitive if for μ -almost every point $x \in X$, the action of the stabilizer $G_x \subset G$ on (X, μ) is ergodic. We say that the action is metrically k -transitive if for almost every (in the sense of the measure μ^k on X^k) choice of points x_1, x_2, \dots, x_{k-1} , the action of the intersection of subgroups $\bigcap_{i=1}^{k-1} G_{x_i}$ on (X, μ) is ergodic.

It is natural to consider this definition only for TNF actions of G . It will be clear that all TNF actions of the infinite symmetric group are k -transitive for an arbitrary positive integer k .

It is of interest to find all countable groups for which TNF k -transitive actions exist for any positive integer k .

This problem is perhaps related to the class of *oligomorphic groups*, which was defined by P. Cameron [3] (a subgroup G of the group $S_{\mathbb{N}}$ of all finite permutations of \mathbb{N} is called oligomorphic if for any positive integer k , the number of orbits of the diagonal action of G in the Cartesian product \mathbb{N}^k is finite).

3 THE LIST OF RANDOM SUBGROUPS OF THE INFINITE SYMMETRIC GROUP

3.1 SIGNED PARTITIONS AND SIGNED YOUNG SUBGROUPS OF SYMMETRIC GROUPS

We consider the countable group $S_{\mathbb{N}}$, the infinite symmetric group of all finite permutations of the set of positive integers \mathbb{N} (or an arbitrary countable set). In this section, we will give the list of all AD-measures on the lattice $L(S_{\mathbb{N}})$ of subgroups of this group and, in particular, the list of TNF measures. We will use some classical facts about permutation groups and the probabilistic approach.

The lattice $L(S_{\mathbb{N}})$ is very large and contains very different types of subgroups. Nevertheless, the support of an AD-measure consists of subgroups of a very special kind: so-called signed Young groups. The topology and the Borel structure on $L(S_{\mathbb{N}})$ are defined as usual; this is a compact (Cantor) space.

Definition 9 (Signed partitions). *A signed partition η of the set \mathbb{N} is a finite or countable partition $\mathbb{N} = \bigcup_{B \in \mathcal{B}} B$ of \mathbb{N} together with a decomposition*

$\mathcal{B} = \mathcal{B}^+ \cup \mathcal{B}^- \cup \mathcal{B}^0$ of the set of its blocks, where \mathcal{B}^0 is the set of all single-point blocks; elements of \mathcal{B}^+ are called positive blocks, and elements of \mathcal{B}^- are called negative blocks (thus each positive or negative block contains at least two points), and we denote by B_0 the union of all single-point blocks: $B_0 = \cup_{\{x\} \in \mathcal{B}^0} \{x\}$.

Denote the set of all signed partitions of \mathbb{N} by $\text{SPart}(\mathbb{N})$.

Recall that in the theory of finite symmetric groups, the Young subgroup Y_η corresponding to an ordinary partition $\eta = \{B_1, B_2, \dots, B_k\}$ is $\prod_{i=1}^k S_{B_i}$, where S_B is the symmetric group acting on B . We will define the more general notion of a *signed Young subgroup*, which makes sense both for finite and infinite symmetric groups. We will use the following notation: $S^+(B)$ is the symmetric group of all finite permutations of elements of a set $B \subset \mathbb{N}$, and $S^-(B)$ is the alternating group on B .⁴

Definition 10 (Signed Young subgroups). *The signed Young subgroup Y_η corresponding to a signed partition η of \mathbb{N} is*

$$Y_\eta = \prod_{B \in \mathcal{B}^+} S^+(B) \times \prod_{B \in \mathcal{B}^-} S^-(B).$$

Note that on the set $B_0 \subset \mathbb{N}$, the subgroup Y_η act identically, so that the partition into the orbits of Y_η coincides with η .

It is not difficult to describe the conjugacy class of Young subgroups with respect to the group of inner automorphisms: $Y_\eta \sim Y_{\eta'}$ if and only if η and η' are equivalent up to the action of $S_{\mathbb{N}}$. But it is more important to consider the conjugacy with respect to the group of outer automorphisms. This is the group $S^{\mathbb{N}}$ of all permutations of \mathbb{N} . Denote by r_0^\pm the number of infinite positive (respectively, negative) blocks, and by r_s^\pm the number of finite positive (respectively, negative) blocks of length $s > 1$. Obviously, the list of numbers $\{r_0^\pm, r_1^\pm, \dots\}$ is a complete set of invariants of the group of outer automorphisms.

⁴Traditionally, the alternating group is denoted by A_n ; V. I. Arnold was very enthusiastic about the idea to denote it by S_n^- in order not to confuse it with the Lie algebra A_n ; I agree with this idea.

3.2 STATEMENT OF THE MAIN RESULT

Consider a sequence of positive numbers $\alpha = \{\alpha_i\}_{i \in \mathbb{Z}}$ such that

$$\alpha_i \geq \alpha_{i+1} \geq 0 \text{ for } i > 0; \quad \alpha_{i+1} \geq \alpha_i \geq 0 \text{ for } i < 0; \quad \alpha_0 \geq 0; \quad \sum_{i \in \mathbb{Z}} \alpha_i = 1.$$

Consider a sequence of \mathbb{Z} -valued independent random variables ξ_n , $n \in \mathbb{N}$, with the distribution

$$\text{Prob}\{\xi_n = v\} = \alpha_v \quad \text{for all } n \in \mathbb{N}, v \in \mathbb{Z}.$$

Thus we have defined a Bernoulli measure μ_α on the space of integer sequences

$$\mathbb{Z}^{\mathbb{N}} = \{\xi = \{\xi_n\}_{n \in \mathbb{N}} : \xi_n \in \mathbb{Z}\}.$$

Definition 11 (A random signed Young subgroup and the measures ν_α). *Fix a sequence $\alpha = \{\alpha_i, i \in \mathbb{Z}\}$, and corresponding Bernoulli measure μ_α ; for each realization of the random sequence $\{\xi_n\}$, $n \in \mathbb{N}$, with the distribution μ_α , define a random signed partition $\eta(\xi)$ of \mathbb{N} as follows:*

$$\eta(\xi) = \{B_i \subset \mathbb{N}, i \in \mathbb{Z}\}, \quad B_i := \{n \in \mathbb{N} : \xi_n = i\},$$

here $\mathcal{B}^+ = \{B_i, i > 0\}$; $\mathcal{B}^- = \{B_i, i < 0\}$, and B_0 is understood as the union of one-point blocks. The correspondence $\xi \mapsto \eta(\xi)$ defines a probability measure on the set $\text{SPart}(\mathbb{N})$ of signed partitions, or random signed partition; the image of the Bernoulli measure μ_α . The correspondence $\xi \mapsto Y_{\eta(\xi)}$ defines a measure, which we denote by ν_α , on the set of signed Young subgroups, i.e., a measure on the lattice $L(S_{\mathbb{N}})$ of subgroups of $S_{\mathbb{N}}$.

Note that all nonempty blocks of the random signed partition $\eta(\xi)$ that consist of more than one point are infinite with ν_α -probability one.

Now we describe the list of all AD and TNF measures for the group $S_{\mathbb{N}}$.

Theorem 1. 1. Every measure ν_α is a Borel ergodic AD-measure on the lattice $L(S_{\mathbb{N}})$; every ergodic probability Borel AD-measure on this lattice coincides with the measure ν_α for some α .

2. The measure ν_α is RTNF-measure for all α , and is TNF-measure if and only if $\alpha_i = 0$ for all $i \leq 0$. So adjoint action of the group $S_{\mathbb{N}}$ on the lattice $L(S_{\mathbb{N}})$ with any AD-measure is TNF-action.

3.3 PROOFS

Proof. 1. The easy part of the proof is to check that the measures ν_α are indeed ergodic AD-measures on $L(S_{\mathbb{N}})$. The invariance follows from the construction, because μ_α , being a Bernoulli measure, is invariant under all permutations of indices. The symmetric, alternating, and identity subgroups of the symmetric groups $S(B)$ are normal, so they are $\text{Ad}(G)$ -invariant. Consequently, the measure ν_α , being the image of μ_α , is $\text{Ad}(G)$ -invariant. The ergodicity with respect to permutations also follows from the ergodicity of the Bernoulli measure.

2. Now suppose that we have an ergodic AD-measure on $L(S_{\mathbb{N}})$. We will filter out, step by step, classes of subgroups of $S_{\mathbb{N}}$ that cannot support any AD-measure, and will finally obtain the class of signed Young groups as the only possible class. Then we will construct all AD-measures on this class.

a) A classical result of the theory of permutation groups asserts that the group $S_{\mathbb{N}}$ has no primitive subgroups except the whole group $S_{\mathbb{N}}$ and the alternating group.⁵ This is a more or less direct corollary of the fundamental estimates obtained by C. Jordan for finite symmetric groups, which were generalized by H. Wielandt [17] (see, e.g., [4, Chapter 8]). Namely, this is a corollary of Jordan's theorem asserting that if a primitive subgroup of S_n has an element with support of size k , then $n < \beta(k)$; a sharp bound on $\beta(k)$ is still unknown.

b) Now consider an AD-measure ν on the set of imprimitive but transitive subgroups of $S_{\mathbb{N}}$. Assume that such a subgroup $H \in L(S_{\mathbb{N}})$ has an invariant partition θ . For the action of H to be transitive, all nontrivial minimal blocks of θ must have the same length $l > 1$, which must be finite (because finite permutations cannot move one infinite block to another one).⁶ Denote by $\theta(H)$ the partition of \mathbb{N} into the minimal blocks for H . The map $H \rightarrow \theta(H)$ associates with ν -almost every imprimitive subgroup a partition into blocks of length $l(H)$; because of the $\text{Ad}(G)$ -ergodicity of ν , this length is the same for ν -almost all subgroups H . Thus the map $H \mapsto \theta(H)$ sends ν to a probability measure on the set of partitions of \mathbb{N} with countably many blocks of the same length $l > 1$, and this measure is invariant with respect to the action of $S_{\mathbb{N}}$ on the space of such partitions. Let us show that there are no such finite

⁵A primitive subgroup is a subgroup that has no nontrivial invariant partitions.

⁶Note that the lengths of all blocks for a given imprimitive group may be either bounded (so-called “almost primitive groups”) or unbounded (“totally imprimitive subgroups”), see [4], but this difference is not important for our purposes.

measures.

Lemma 1. *There are no probability measures on the space $\text{Part}(l)$ of all partitions of \mathbb{N} into (countably many) blocks of length $l > 1$ that are invariant with respect to the group $S_{\mathbb{N}}$.*

Remark. The space $\text{Part}(l)$ equipped with the weak topology is locally compact but not compact; its natural compactification consists of all partitions whose blocks have length at most l .

Proof. Consider the case $l = 2$, the same proof is true for an arbitrary l . Each partition from $\text{Part}(2)$ determines a symmetric matrix (for $l > 2$ -symmetric tensor) $\{a_{i,j}\}$, $a_{i,j} = a_{j,i}$, $a_{i,i} = 0$, $i, j = 1, 2, \dots$, with only one entry in each row and each column equal to 1, all the other entries being equal to 0. But because of the $S_{\mathbb{N}}$ -invariance, we have a random symmetric matrix $\{a_{i,j}\}$, unique element in each row which is equal to 1 must be uniformly distributed along its row. It is impossible for infinite matrix. \square

c) We have proved that an AD-measure on the lattice $L(S_{\mathbb{N}})$ takes the value 0 on the set of all transitive subgroups; so we have reduced the analysis to intransitive subgroups. Fix such a generic intransitive subgroup $H \in L(S_{\mathbb{N}})$ and consider the maximal partition $\eta(H)$ into its transitive components. The action of the group H on each such component must be primitive, because imprimitive cases can be discarded for the same reason as in the previous part of the proof. For the same reason, it is obvious that all components of this partition must be infinite. Consequently, the action of H on each component is either the action of the whole symmetric group, or that of the alternating group (see part a) of the proof), or that of the identity group on the single-point blocks. We denote the blocks by B_i , $i > 0$, when the action of H is the action of the symmetric group of B_i , and by B_i , $i < 0$, when the action of H is the action of the alternating group of B_i . The action of the identity group on all single-point blocks can be regarded as the identity action on the union of such blocks B_0 . Thus we have a *signed partition* $\eta(H)$ such that the action of H on each block B_i , $i > 0$, is the action of $S^+(B)$, the action of H on each block B_i , $i < 0$, is the action of $S^-(B)$, and the action on B_0 is the identity action. This means that $H \subset Y_{\eta} = \prod_i S^{\pm}(B_i)$ and the restriction of the action of H to B_i is the action of $S^{\pm}(B)$.

For each $i \neq 0$ denote the group

$$K_i = \{g \in S^{\pm}(B_i) : \exists \bar{g} \in H, \quad \bar{g}|_{B_j} = id, \quad \forall j \neq i, \quad \bar{g}|_{B_i} = g\}$$

or a subgroup of the all elements in H which acts as identity on all $B_j, j \neq i$.

It is clear that K_i is a normal subgroup in $S^\pm(B_i)$ (because it is the kernel of homomorphism), so K_i is either $S^\pm(B_i)$ or $K_i = \{id\}$, and

$$\prod_{i \neq 0} K_i \subset H$$

Thus we need to prove that $K_i = H|_{B_i} = S^\pm(B_i)$ for all $i \neq 0$ (and in particular $K_i \neq \{id\}$ if $i \neq 0$). There are no problem with i if $K_i = S^+(B_i) = H|_{B_i}$. We must consider two cases: the first case when $K_i = \{id\}$ but $H|_{B_i} = S^\pm(B_i)$ (in this case it does not matter $H|_{B_i} = S^+$ or S^- , so $i \neq 0$), and the second case when $K_i = S^-(B_i) \neq H|_{B_i} = S^+(B_i)$. Let us consider the first case. Suppose for some $i \neq 0$ $K_i = \{id\}$ but $H|_{B_i} = S^\pm(B_i)$. Then there exist at least one $j \neq i$ for which

$$K_i = \{g \in S^\pm(B_i) : \exists \bar{g} \in H, \quad \bar{g}|_{B_j} = id, \quad \bar{g}|_{B_i} = g\},$$

indeed the intersection could be either $\{id\}$ or $S^\pm(B_i)$ for all j and if the intersection in the definition of K_i over all $j \neq i$ is $\{id\} = K_i$, then such j exists. It means that for this j and for $h \neq id, h \in H|_{B_i}$ there exists $h' \in H|_{B_j}$ and $g \in H$ such that $g|_{B_i} = h, g|_{B_j} = h'$. So we have a map from $H|_{B_i}$ to $H|_{B_j}$ which is homomorphism, and consequently isomorphism which is simply bijection - T - between B_i and B_j . This bijection could be arbitrary because of invariance under conjugation of the group. Thus the action on of the group H on $B_i \cup B_j$ is as follow: if $n \in B_i$ and $Tn = m \in B_i$, then $gm = Tgn$. or $gT = Tg$ on $B_i \cup B_j$. If we restrict the action of H on $B_i \cup B_j$ only, we obtain that the group H acts periodically (or "simultaneously") on B_i and B_j .

Lemma 2. *There are no AD-invariant measures which are concentrated on the intransitive subgroups $H \subset S_{\mathbb{N}}$ of the following type: If $\mathbb{N} = \mathbb{N}' \times K$, (\mathbb{N}' is infinite), then*

$$H = S_{\mathbb{N}'} \times \{id_K\} \subset S_{\mathbb{N}} \quad H = \{g : g = (g', id_K); g' \in S_{\mathbb{N}'}\},$$

or periodic action on $\mathbb{N}' \times K$.

Proof. The random group H of this type must define a $S_{\mathbb{N}}$ -invariant random partition of \mathbb{N} onto $|K|$ parts and $S_{\mathbb{N}'}$ -invariant random bijections between all parts. The invariant random partitions do exist -see the next item but invariant bijection do not because the absence of probability measure on the group $S_{\mathbb{N}'}$. \square

So we don't need to consider the subgroups H for which the first case takes place and consequently we already proved that $K_i = S^\pm(B_i)$ $i \neq 0$ (we write S^\pm when it is not important either S^+ or S^-).

Suppose now that for some i , $K_i \supset S^-(B_i)$, and $H|_{B_i} = S^+(B_i)$. Again find $j \neq 0$ for which $K_i = \{g \in S^+(B_i) : \exists \bar{g} \in H, \bar{g}|_{B_j} = id, \bar{g}|_{B_i} = g\}$. Because of definition of K_i it is clear that $H|_{B_i \cup B_j} \supset S^-(B_i) \times S^\pm(B_j)$, and the last subgroup has index in $H|_{B_i \cup B_j}$ at most two, but also we have $H|_{B_i} = S^+(B_i)$, so $H|_{B_i \cup B_j} = S^+(B_i) \times S^\pm(B_j)$. But this means that $K_i = H|_{B_i} = S^+(B_i)$.

So we prove that $H = \prod_i H|_{B_i}$ and each $H|_{B_i} = S^\pm(B_i)$ for $i \neq 0$, or in another words we have proved that H must be a signed Young subgroups: only signed Young subgroups can carry AD-invariant measures on the lattice $L(S_{\mathbb{N}})$.

The measures ν_α which was defined above are concentrated on the signed Young subgroup by definition.

d) Now we will prove that indeed this case is realized: the random subgroups in the infinite symmetric group or AD-invariant ergodic measure on $L(S_{\mathbb{N}})$ is one of the measure ν_α and indeed each measure ν_α are AD-invariant ergodic measure on $L(S_{\mathbb{N}})$.

We must identify the required measures with the ergodic limits with respect to conjugation of signed Young subgroups. Because of the correspondence between signed Young subgroups and signed partitions, this question is equivalent to the description of $S_{\mathbb{N}}$ -invariant measures on the set of signed partitions. The last question is similar to the classical de Finetti problem concerning $S_{\mathbb{N}}$ -invariant measures on the space of all functions on \mathbb{N} (see [10]). The only small difference lies in the fact that, in contrast to the classical situation, here we have three types of blocks of signed partitions instead of one type in the ordinary de Finetti theorem.

Lemma 3 (An analog of classical de Finetti's theorem; Kingman's theorem [13]). *Consider the space $\text{SPart}(\mathbb{N})$ of signed partitions of \mathbb{N} ; every ergodic $S_{\mathbb{N}}$ -invariant measure on $\text{SPart}(\mathbb{N})$ is determined by a sequence $\alpha = \{\alpha_i\}_{i \in \mathbb{Z}}$ such that $\alpha_i \geq \alpha_{i+1} \geq 0$ for $i > 0$, $\alpha_{i+1} \geq \alpha_i \geq 0$ for $i < 0$, $\alpha_0 \geq 0$, and $\sum_{i \in \mathbb{Z}} \alpha_i = 1$, as described above.*

Proof. The lemma can be proved by any of the methods people use to prove de Finetti's theorem. For completeness, we present a proof, applying our old ergodic method from [10]. In order to find all ergodic measures ν on a compact $S_{\mathbb{N}}$ -space X using the pointwise ergodic theorem for the group $S_{\mathbb{N}}$

(which is in fact a theorem on the convergence of martingales), it suffices to find the weak limits of measures (when they do exist)

$$\lim \frac{1}{n!} \sum_{g \in S_n} \delta_{gx}$$

for all $x \in X$. More exactly, we need to calculate the limits

$$\lim_n \frac{1}{n!} \sum_{g \in S_n} f(gx)$$

for continuous functions $f \in C(X)$. In our case, it suffices to consider cylinder functions on $\text{SPart}(\mathbb{N})$ which depend on finitely many blocks. Fix a signed partition η and label its blocks with integers in an arbitrary way so that positive (negative) integers correspond to positive (negative) blocks and B_0 is the union of one-point blocks. Consider the \mathbb{Z} -valued sequence x_n , $n \in \mathbb{N}$, defined as follows: $x_n = s$ if $n \in B_s$. Now we may say that the signed partition η is the partition corresponding to the sequence $\{x_n\}$, and each such sequence determines a signed partition. The action of $S_{\mathbb{N}}$ on the set of signed partitions and its action by permutations of coordinates of sequences obviously agree, so our problem reduces to the description of $S_{\mathbb{N}}$ -invariant measures on the space of all elements of $\mathbb{Z}^{\mathbb{N}}$. But this is exactly de Finetti's problem. Start with an arbitrary sequence $\{x_n\} \in \mathbb{Z}^{\mathbb{N}}$ and calculate the limit

$$\lim_n \frac{1}{n!} \#\{g \in S_n : x_{gn} = v\} = \alpha_v$$

under the assumption that it does exist (it exists for almost all $x \in X$). Thus we obtain the one-dimensional distribution of the random (with respect to the measure ν) sequence x_n . In order to prove that this measure is a Bernoulli measure on $\mathbb{Z}^{\mathbb{N}}$, we must calculate the joint distribution of several coordinates of x_n . But because of the complete transitivity of the action of S_n , for any choice of v_1, \dots, v_t and for $n \gg t$ we have

$$\frac{\#\{g \in S_n : x_{gi} = v_i, i = 1, 2, \dots, t, n \gg t\}}{n!} \approx \prod_{i=1}^t \alpha_{v_i},$$

which means that the random variables x_n , $n = 1, 2, \dots$, are independent. Thus all AD-measures arise from Bernoulli measures on the space of signed partitions, i.e., $\nu = \nu_{\alpha}$ for some α . The assertion of the theorem for the unsigned partitions is Kingman's theorem ([13]), but our proof is different. \square

e) Consider the random signed Young subgroup Y_η constructed from a sequence α with $\alpha_i = 0$ for all $i \leq 0$. Then all blocks B_i , $i \leq 0$, are empty with probability one. Then, obviously, the normalizer $N(Y_\eta)$ coincides with Y_η , since each block of η gives rise to the self-normalizing subgroup $S^+(B_i)$. Consequently, the measure ν_α is TNF. If $\alpha_i > 0$ for some $i \leq 0$, then the corresponding block B_i is not empty with probability one, whence $N(S_{B_i}^-) = S_{B_i}^+ \neq S_{B_i}^-$, so that Y_η is not self-normalizing. But

$$N(Y_\eta) = \prod_{i=-\infty}^{+\infty} S_{B_i}^+.$$

Thus $N^2(Y_\eta) = N(Y_\eta)$, so that ν_α is a RTNF measure in the terminology of Section 1. This completes the proof of Theorem 1. \square

Corollary 1. *The action of the group $S_{\mathbb{N}}$ on the measure space $(L(S_{\mathbb{N}}), \nu_\alpha)$ is ergodic.*

Indeed, this is a corollary of the fact that the measure ν_α is the image of the Bernoulli measure μ_α and the correspondence $\mu_\alpha \mapsto \nu_\alpha$ between measures commutes with the action of the group. The corresponding representation of the group $S_{\mathbb{N}}$ in the space $L^2_{\nu_\alpha}(L(S_{\mathbb{N}}))$ will be considered elsewhere.

Corollary 2. *There are three degenerate measures ν_α , in the following cases (in the parentheses we indicate the corresponding character and representation, see below):*

1) $\alpha_1 = 1$, $\alpha_i = 0$, $i \neq 1$; in this case, $\nu_\alpha = \delta_{S_{\mathbb{N}}} (\chi(g) \equiv 1, \text{ the identity representation})$;

2) $\alpha_{-1} = 1$, $\alpha_i = 0$, $i \neq -1$; in this case, $\nu_\alpha = \delta_{S_{\mathbb{N}}}^- (\chi(g) = (-1)^{\text{sgn}(g)}, \text{ the alternating representation})$;

3) $\alpha_0 = 1$, $\alpha_i = 0$, $i \neq 0$; in this case, $\nu_\alpha = \delta_{\text{Id}_{S_{\mathbb{N}}}} (\chi(g) = \delta_e(g), \text{ the regular representation})$.

An ergodic AD-measure ν_α is atomic only in these three cases (in which it is in fact a δ -measure); in all the other cases, ν_α is a continuous measure.

3.4 REMARKS AND A FORMULA FOR THE MEASURES OF THE SETS OF FIXED POINTS

Make sense to compare the language which we use here (the action on $L(G)$) with that which was used in [16] (the action on the Bernoulli sequences).

More concretely, consider the action of $S_{\mathbb{N}}$ on the space $\mathbb{Z}^{\mathbb{N}}$ (instead of $L(S_{\mathbb{N}})$) and ask for a description of TNF and RTNF measures. The answer is a little bit different than for the action on the space of Young subgroups. Namely, the following result holds.

Proposition 7. *The measure μ_{α} on the space $X = \mathbb{Z}^{\mathbb{N}}$ with the action of the group $S_{\mathbb{N}}$ is a TNF measure if and only if all α_i , $i \neq 0$, are distinct. If $\alpha_i = \alpha_j$ for some $i \neq j$, then the action of $S_{\mathbb{N}}$ is RTNF but not TNF. The canonical projection $X \rightarrow X/\xi_G$ is the factorization with respect to the following equivalence relation on $X = \mathbb{Z}^{\mathbb{N}}$: two elements $\{x_n\}_{n \in \mathbb{N}}, \{x'_n\}_{n \in \mathbb{N}} \in X$ are equivalent for if for every $v \in \mathbb{Z}$ either $\{n \in \mathbb{N} : x_n = v\} = \{n \in \mathbb{N} : x'_n = v\}$, or there exists $v' \in \mathbb{Z}$ with $\alpha_v = \alpha_{v'}$ such that*

$$\{n \in \mathbb{N} : x_n = v\} = \{n \in \mathbb{N} : x'_n = v'\},$$

and

$$\{n \in \mathbb{N} : x_n = v'\} = \{n \in \mathbb{N} : x'_n = v\}.$$

Thus, in this case the action is RTNF not TNF if we have multiplicity in the values of α : $\alpha_i = \alpha_j, i \neq j$; this is not the case for the action in $L(S_{\mathbb{N}})$.

The supports of the measures ν_{α} in the topological sense (i.e., the minimal closed subsets of full measure) coincide for all α that have the same number of infinite blocks. The support of ν_{α} for α having infinitely many infinite blocks coincides with the space of all signed Young subgroups.

In the case of the infinite symmetric group, all AD-measures are invariant under the group $S^{\mathbb{N}}$ of all permutations of \mathbb{N} . The conjugation with respect to this group is an extension of the usual conjugation; but for a generic subgroup H from a set A of full ν_{α} -measure, its orbit under the action of $S^{\mathbb{N}}$ is much larger than A . In other words, the frequencies α_i are invariant under the usual conjugation, but not under its extension. This fact is related to the so-called Kolmogorov effect (see [11]).

Theorem 1 gives more than just the list of AD-measures on the group $S_{\mathbb{N}}$; it helps to give a new proof of Thoma's formula for indecomposable characters of this group. This will be the subject of our next article, and now we merely carry out the calculations and give a short commentary. Here we present the formula for characters in the "positive" case.⁷

⁷For convenience, we have slightly changed the notation: usually, $\alpha_i \equiv \beta_i$ for $i < 0$, and $\alpha_0 \equiv \gamma$.

Theorem 2. *For an ergodic AD-measure ν_α ,*

$$\nu_\alpha(F_g) \equiv \nu_\alpha\{H : gHg^{-1} = H\} = \nu_\alpha\{H : g \in N(H)\} = \prod_{n>1} [p_n(\alpha)]^{c_n(g)},$$

where

$$p_n(\alpha) = \sum_{i \neq 0} \alpha_i^n$$

(Newton's power sum) and $c_n(g)$ is the number of cycles of length $n > 1$ of a permutation g .

In the case where $\alpha_i = 0$ for $i < 0$, this formula coincides with Thoma's formula [15] for characters of the infinite symmetric group, because the measure of the set of fixed points is equal to the value of the character:

$$\chi_\alpha(g) = \nu_\alpha(\text{Fix}(g)).$$

In the general case, Thoma's formula involves super-Newton instead of Newton sums:

$$p_n(\alpha) = \sum_{i>0} \alpha_i^n + (-1)^{n-1} \sum_{i<0} \alpha_i^n = \sum_{i \neq 0} (\text{sgn } i)^{n-1} \alpha_i^n.$$

The measure of the set of fixed points does not depend on the types of blocks, but for a general parameter α , the value of the character is not equal just to the measure of this set, the formula involving a certain multiplier (see [16]). We will return to this question and give a model of representations in the next article.

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